

# Sums of free variables in fully symmetric spaces

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## ABSTRACT

We give a method to obtain, from Voiculescu’s inequality, norm estimates for sums of free variables with amalgamation in general fully symmetric spaces. We use these estimates to interpolate the Burkholder inequalities for non-commutative martingales. The method is also applicable to other similar settings. In that spirit, we improve known results on the non-commutative Johnson–Schechtman inequalities and recover Khinchin inequalities associated to free groups.

## 1. Introduction

Versions of Khinchin inequalities for Schatten classes of index  $1 < p < \infty$  were established by Lust–Piquard in the mid 1980s. They are one of the first evidence of a new non-commutative phenomenon; one has to deal with different notions of square functions in quantum analysis. Since then, they were omnipresent in all the developments of non-commutative analysis. They become part of the theory and are used to define the right function spaces. For instance, the formulation of the Burkholder–Gundy inequalities for martingales led to the definition of the column Hardy space  $\mathcal{H}_p^c(\mathcal{M})$  and its row version  $\mathcal{H}_p^r(\mathcal{M})$  associated to a semi-finite von Neumann algebra. The martingale Hardy spaces are then defined as  $\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) \cap \mathcal{H}_p^r(\mathcal{M})$  for  $2 < p < \infty$  and  $\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) + \mathcal{H}_p^r(\mathcal{M})$  for  $1 \leq p < 2$ .

One of the main drawback is the difficulty to understand the behaviour of those square functions with respect to interpolation theory. Indeed, it is not clear how to deal with intersections or sums of two spaces in full generality. Since there is only one square function when the underlying algebra is commutative, those problems do not occur at all. Much efforts have been made to study the interpolation of non-commutative  $L_p$ -inequalities in various contexts, for instance, [4–6, 8–10, 21, 25]. Given a function space  $E$ , say on  $(0, \infty)$ , one can associate a non-commutative space  $E(\mathcal{M})$  to any semi-finite von Neumann algebra [15]. The general question is the following: given a function space  $E$  which is an interpolation space for  $(L_p, L_q)$  and knowing an inequality that is true for  $L_p(\mathcal{M})$  and  $L_q(\mathcal{M})$ , can we get a new one for  $E(\mathcal{M})$ ? Most of the papers quoted above relied on very elaborated machineries on function spaces and quite satisfactory results are available but under technical conditions (such as on Boyd indices).

On the opposite side, freeness in quantum probabilities behaves in a nicer way than independence in classical probabilities. The central limit object, a semi-circular variable, is bounded in  $L_\infty$ . Many inequalities still hold true when  $p = \infty$ , the most famous example is the Haagerup inequality for generators of free groups (which is seen as a Khinchin type inequality). It has been a key tool to deduce interpolation results for intersections or sums of spaces coming from square functions [18]. Very recently in [4], the first author discovered a very efficient way to deal with interpolation of the non-commutative Khinchin inequalities. The main novelty is

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that one can use certain algebraic decompositions to play the role of square functions. With the help of freeness, it was used to interpolate the Burkholder–Gundy inequalities. This paper is an attempt to show that this technique is fairly general and can be used to overcome quite easily the problems of interpolation of non-commutative function spaces, basically without any assumption.

After the Khinchin inequality, the next interesting mixed-norms are given by the so-called Voiculescu inequality which is a combination of three different norms in the spirit of the Rosenthal inequality for sums of independent variables. They appeared in [11, 12] and are hidden in the conditioned version of the Burkholder–Gundy inequality. To describe them, let  $\mathcal{N} \subset \mathcal{M}$  be finite von Neumann algebras with a normal faithful trace  $\tau$  and a trace-preserving conditional expectation  $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$ . For a family  $(x_i)_{i \geq 1}$  and  $1 \leq p \leq \infty$ , the three norms involved are

$$\begin{aligned} \|(x_i)\|_{p,c} &= \left\| \left( \mathcal{E} \sum_{i=1}^{\infty} x_i^* x_i \right)^{1/2} \right\|_{L_p(\mathcal{M})}, \quad \|(x_i)\|_{p,r} = \|(x_i^*)\|_{p,c}, \\ \|(x_i)\|_{p,d} &= \left( \sum_{i=1}^{\infty} \|x_i\|_{L_p(\mathcal{M})}^p \right)^{1/p}. \end{aligned}$$

It was established in [12] for  $p = \infty$  and in [13] for  $2 < p < \infty$ , that if the variables  $(x_i)_{i \geq 1}$  are free and centered in  $\mathcal{M}$  over  $\mathcal{N}$ , then for some constants independent of  $p$

$$\left\| \sum_{i=1}^{\infty} x_i \right\|_{L_p(\mathcal{M})} \approx \|(x_i)\|_{p,c} + \|(x_i)\|_{p,r} + \|(x_i)\|_{p,d}.$$

One can deduce a statement for  $1 \leq p < 2$  by duality using an infimum as usual. One of our objective is to interpolate those inequalities. If  $E$  is a symmetric space, we set

$$\begin{aligned} \|(x_i)\|_{E,c} &= \left\| \left( \mathcal{E} \sum_{i=1}^{\infty} x_i^* x_i \right)^{1/2} \right\|_{E(\mathcal{M})}, \quad \|(x_i)\|_{E,r} = \|(x_i^*)\|_{E,c}, \\ \|(x_i)\|_{E,d} &= \left\| \sum_{i=1}^{\infty} x_i \otimes e_i \right\|_{E(\mathcal{M} \overline{\otimes} \ell_{\infty})}, \end{aligned}$$

where  $(e_i)$  is the canonical basis of  $\ell_{\infty}$  equipped with its standard trace. We obtain that if  $E$  is an interpolation space for  $(L_2, L_{\infty})$  and  $\sum_{i=1}^{\infty} x_i \in \mathcal{M}$ ,

$$\left\| \sum_{i=1}^{\infty} x_i \right\|_{E(\mathcal{M})} \approx \|(x_i)\|_{E,c} + \|(x_i)\|_{E,r} + \|(x_i)\|_{E,d}.$$

Similarly when  $E$  is an interpolation space for  $(L_1, L_2)$

$$\left\| \sum_{i=1}^{\infty} x_i \right\|_{E(\mathcal{M})} \approx \inf_{x_i = a_i + b_i + d_i} \|(a_i)\|_{E,c} + \|(b_i)\|_{E,r} + \|(c_i)\|_{E,d},$$

one can even choose  $a_i, b_i, c_i$  independently of  $E$ . The proof of the main inequalities follows three steps. First, we show that the above infimum is achieved when  $E = L_1$  using a compactness argument. A sequence attaining the minimum is called an optimal decomposition. We then show that an optimal decomposition has a certain algebraic form which is used in the last step to deduce the inequalities from that for  $E = L_{\infty}$ . This is a fairly general principle.

In the last section, we sketch a proof of a similar interpolation result around the Haagerup–Buchholz inequality [3, 19] for words of length  $d$  using a generating set in the free group algebra. We believe that the techniques could be pushed in many other directions like the general Rosenthal type inequality for free chaos of [13] which is technically more involved; we leave it as a problem for the interested reader.

We also relate our results with the Johnson–Schechtman inequalities for free variables obtained in [25]. They deal with the simplest case  $\mathcal{N} = \mathbb{C}$ . As their commutative counterpart, they extend Rosenthal type inequalities to some symmetric spaces. They give a fully computable expression for  $\|\sum_{i=1}^{\infty} x_i\|_{E(\mathcal{M})}$ . In this case, an algebraic decomposition can be given explicitly, this leads to an improvement of the constants in [25]. We also explain how to use our main result to interpolate the conditioned Burkholder inequality in the spirit of [5, 21]. After we submitted this article, Randrianantoanina informed us that he also obtained with Xu these martingale inequalities in [22] but with different techniques.

## 2. Preliminaries

### 2.1. Non-commutative integration

We use [18, 20, 26] and [7] as general references for non-commutative integration in the semi-finite setting. We use the classical definition of the non-commutative  $L_p$ ,  $1 \leq p < \infty$  associated to a semi-finite von Neumann algebra  $(\mathcal{M}, \tau)$

$$L_p(\mathcal{M}, \tau) = \{x \in L_0(\mathcal{M}, \tau) \mid \|x\|_p^p = \tau(|x|^p) < \infty\},$$

where  $L_0(\mathcal{M}, \tau)$  is the space of  $\tau$ -measurable operators affiliated with  $\mathcal{M}$  (see [26]). We also set as usual  $L_\infty(\mathcal{M}) = \mathcal{M}$  with its standard norm. Of course we always have that  $L_p(\mathcal{M}, \tau) \subset L_q(\mathcal{M}, \tau)$  whenever  $q \leq p$  when  $\mathcal{M}$  is finite ( $\tau(1) < \infty$ ).

Given  $x \in L_0(\mathcal{M}, \tau)$ , its generalized singular values [7] is a function  $\mu(x, \tau) : (0, \infty) \rightarrow (0, \infty)$  which is non-increasing and has the same distribution as  $x$  when  $(0, \infty)$  is equipped with the standard Lebesgue measure. We may drop the reference to  $\tau$  when it is not necessary.

A Banach function space  $(E, \|\cdot\|_E)$  on  $(0, \infty)$  is said to be symmetric if it is included in  $L_0(L_\infty(0, \infty)) = L_0(0, \infty)$  and if whenever  $f \in E$  and  $g \in L_0(0, \infty)$  satisfy  $\mu(g) = \mu(f)$ , then  $g \in E$  and  $\|g\|_E = \|f\|_E$ . The non-commutative version of  $E$  associated to  $(\mathcal{M}, \tau)$  is the space  $E(\mathcal{M}, \tau) = \{x \in L_0(\mathcal{M}, \tau) \mid \mu(x) \in E\}$  with the norm  $\|x\|_{E(\mathcal{M})} = \|\mu(x)\|_E$ , see [15]. Note that if  $(\mathcal{N}, \tau) \subset (\mathcal{M}, \tau)$  is a von Neumann subalgebra, then for  $y \in L_0(\mathcal{N}, \tau)$ ,  $\|y\|_{E(\mathcal{N})} = \|y\|_{E(\mathcal{M})}$ .

We will focus on fully symmetric function spaces. These are symmetric spaces  $E$  such that if  $f \in E$  and  $g \in L_0(0, \infty)$  satisfy for all  $t > 0$ ,  $\int_0^t \mu(g) \leq \int_0^t \mu(f)$ , then  $g \in E$  and  $\|g\|_E \leq \|f\|_E$ . They admit several other characterizations especially using general interpolation, we refer to [1, 2]. This is related to the fact that  $\int_0^t \mu(f) = \|f\|_{L_1 + tL_\infty}$  for  $t > 0$  and  $f \in L_0(0, \infty)$ .

An interpolation space for the couple  $(L_p(0, \infty), L_q(0, \infty))$ ,  $1 \leq p, q \leq \infty$  is a Banach function space  $E \subset L_p(0, \infty) + L_q(0, \infty)$  such that if  $T : L_p(0, \infty) + L_q(0, \infty) \rightarrow L_p(0, \infty) + L_q(0, \infty)$  is a linear map such that  $\|T\|_{L_p(0, \infty) \rightarrow L_p(0, \infty)}, \|T\|_{L_q(0, \infty) \rightarrow L_q(0, \infty)} \leq 1$ , then  $T(E) \subset E$  and  $\|T\|_{E \rightarrow E} \leq 1$ . Fully symmetric spaces are exactly interpolation spaces for the couple  $(L_1(0, \infty), L_\infty(0, \infty))$ .

When  $E$  is a fully symmetric function space on  $(0, \infty)$  or more specifically an interpolation space for  $(L_p, L_q)$ , the non-commutative function spaces associated to it also enjoy the same properties, see [20, Corollary 2.2]. We will mainly use that when  $E$  is an interpolation for  $(L_p, L_q)$  with  $p \leq q$  if  $(\mathcal{M}_1, \tau_{\mathcal{M}_1})$  and  $(\mathcal{M}_2, \tau_{\mathcal{M}_2})$  are semi-finite von Neumann algebras and if  $T : L_p(\mathcal{M}_1, \tau_{\mathcal{M}_1}) \rightarrow L_p(\mathcal{M}_2, \tau_{\mathcal{M}_2})$  is a map that is contractive on  $L_p$  and  $L_q$ , that is,  $\|T\|_{L_r(\mathcal{M}_1, \tau_{\mathcal{M}_1}) \rightarrow L_r(\mathcal{M}_2, \tau_{\mathcal{M}_2})} \leq 1$  for  $r = p, q$ , then  $T(E(\mathcal{M}_1)) \subset E(\mathcal{M}_2)$  and  $T$  is a contraction from  $E(\mathcal{M}_1)$  to  $E(\mathcal{M}_2)$ . We also refer to [15] for more on this topic.

In the whole paper, we always consider fully symmetric spaces over  $(0, \infty)$ , this is not a restriction (see Remark 4.7). The von Neumann algebras  $(\mathcal{M}, \tau)$  will always be non-commutative probability spaces ( $\tau(1) = 1$ ) except  $\mathcal{M} \overline{\otimes} \ell_\infty$  with its natural trace and  $L_\infty(0, \infty)$ .

## 2.2. Free products

We fix  $N \in \mathbb{N}^* \cup \{\infty\}$ . If  $(\mathcal{M}_i, \tau_i)$ ,  $1 \leq i \leq N$  are finite von Neumann algebras with a common sub-von Neumann algebra  $(\mathcal{N}, \tau)$  and conditional expectations  $\mathcal{E}_i : \mathcal{M}_i \rightarrow \mathcal{N}$  such that  $\tau \circ \mathcal{E}_i = \tau_i$ , we denote by  $(\mathcal{M}, \tau) = *_{i=1 \dots N} (\mathcal{M}_i, \tau_i)$  the amalgamated free product of the algebras  $(\mathcal{M}_i, \tau_i)$  over  $\mathcal{N}$ . We refer to [28] for precise definitions. We simply recall basic facts. If  $x \in \mathcal{M}_i$ , we denote by  $\hat{x} = x - \mathcal{E}_i x$  and  $\mathring{\mathcal{M}}_i = \{\hat{x}; x \in \mathcal{M}_i\}$ ; there is a natural decomposition  $\mathcal{M}_i = \mathcal{N} \oplus \mathring{\mathcal{M}}_i$ . The space  $W = \mathcal{N} \oplus_{n \geq 1} \bigoplus_{1 \leq i_1 \neq i_2 \neq \dots \neq i_n \leq N} \mathring{\mathcal{M}}_{i_1} \otimes \mathcal{N} \dots \otimes \mathring{\mathcal{M}}_{i_n}$  is a  $*$ -algebra where the product is given by concatenation and centering with respect to  $\mathcal{N}$ . It has a trace given by  $\tau_{\mathcal{N}} \circ \mathcal{E}$  where  $\mathcal{E}$  is the natural projection onto  $\mathcal{N}$ . Then  $(\mathcal{M}, \tau)$  is the finite von Neumann algebra obtained by the GNS construction from  $(W, \tau)$ . Elements in  $\bigoplus_{1 \leq i_1 \neq i_2 \neq \dots \neq i_n \leq N} \mathring{\mathcal{M}}_{i_1} \otimes \mathcal{N} \dots \otimes \mathring{\mathcal{M}}_{i_n}$  are said to be of length  $n$ .

To lighten notations, seeing  $(\mathcal{M}_i, \tau_i)$  as a sub-von Neumann algebra of  $(\mathcal{M}, \tau)$ , we have that  $\tau|_{\mathcal{M}_i} = \tau_i$  and  $\mathcal{E}|_{\mathcal{M}_i} = \mathcal{E}_i$  and we will simply write  $\tau$  and  $\mathcal{E}$  instead of  $\tau_i$  and  $\mathcal{E}_i$ .

We use the notation  $\tilde{y}$  for  $y - \mathcal{E}y$  for  $y \in L_1(\mathcal{M})$ .

## 2.3. Column conditioned norms

In this section, we assume that  $(\mathcal{M}, \tau)$  is a finite von Neumann algebra with a subalgebra  $\mathcal{N}$ . The trace  $\tau$  is well defined on  $L_0(\mathcal{M})^+$  by  $\tau(x) = \sup_n \tau(x1_{[0, n]}(x))$  (see [26]). In particular, for any  $x \in L_0(\mathcal{M})$ ,  $\tau(x^*x) = \tau(xx^*)$  and if  $q_n$  is a non-decreasing sequence of projections going to 1 strongly and  $x \in L_0(\mathcal{M})$ , then  $\tau(x^*x) = \sup_n \tau(x^*q_nx)$  (this is obvious when  $x \in L_2$ ).

For  $x \in L_2(\mathcal{M})$ ,  $\mathcal{E}x^*x$  is well defined in  $L_1(\mathcal{N}) \subset L_1(\mathcal{M})$ .

DEFINITION 2.1. For  $x \in L_2(\mathcal{M})$  and  $1 \leq p \leq \infty$ , we set  $\|x\|_{p,c} = \|(\mathcal{E}x^*x)^{1/2}\|_p$ .

Note that  $\|x\|_{p,c} < \infty$  when  $1 \leq p \leq 2$  and  $\|x\|_{2,c} = \|x\|_2$ .

The completion of the set of elements  $x$  satisfying  $\|x\|_{p,c} < \infty$  is denoted by  $L_p(\mathcal{M}, \mathcal{E})$  in [11]. Most of the results in this section can be collected from that paper or [12]. But for completeness of the next section, we will give a different proof of the basic facts we need; [11, 12] also deal with type III von Neumann algebras that we do not consider.

LEMMA 2.2 ( $\mathcal{E}$ -polar decomposition). Let  $E$  be right- $\mathcal{N}$ -submodule of  $L_2(\mathcal{M})$ . Let  $x \in E$ , then there exists  $u \in L_2(\mathcal{M})$  such that  $\|u\|_{\infty,c} \leq 1$  and  $x = u(\mathcal{E}x^*x)^{1/2}$ .

Moreover,  $s(\mathcal{E}x^*x) \leq \mathcal{E}u^*u \leq 1$  and  $u1_{[\varepsilon, \infty)}(\mathcal{E}x^*x) \in E$  for all  $\varepsilon > 0$ .

Proof. Let  $p$  be the support of  $\alpha = (\mathcal{E}x^*x)^{1/2}$  in  $\mathcal{N}$ .

First note that  $x = xp$  as  $\tau((1-p)x^*x(1-p)) = \tau((1-p)\alpha^2) = 0$ . Then the element  $y = \alpha^{-1} \in L_0(\mathcal{N}) \subset L_0(\mathcal{M})$  and  $u = xy$  is well defined in  $L_0(\mathcal{M})$ . Clearly  $u\alpha = xp = x$  in  $L_0(\mathcal{M})$ .

Let us check that  $u \in L_2(\mathcal{M})$ . Set  $p_n = 1_{\{0\} \cup [\frac{1}{n}, n]}(\alpha)$ , then  $p_n \nearrow 1$  and by normality of  $\tau$ ,

$$\tau(u^*u) = \tau(uu^*) = \lim_n \tau(xyp_nyx^*) = \lim_n \tau(p_nyx^*xyp_n).$$

Since  $p_ny \in \mathcal{N}$ , by the modular property of conditional expectations

$$\tau(p_nyx^*xyp_n) = \tau(p_ny\alpha^2yp_n) \leq 1.$$

We also have that  $u1_{[\varepsilon, \infty)}(\alpha) = x1_{[\varepsilon, \infty)}(\alpha)\alpha^{-1} \in E$  for all  $\varepsilon > 0$  as  $1_{[\varepsilon, \infty)}(\alpha)\alpha^{-1} \in \mathcal{N}$ .

Next,

$$\|u\|_{\infty,c}^2 = \|\mathcal{E}u^*u\|_\infty = \sup_z \tau\left(z^*(\mathcal{E}u^*u)z\right) = \sup_z \|uz\|_2^2,$$

where  $z$  runs over all elements in  $\mathcal{N}$  with  $\|z\|_2 \leq 1$ . As  $u \in L_2(\mathcal{M})$  and  $z \in \mathcal{N}$ ,  $\|uz\|_2 = \lim_n \|up_n z\|_2$  so that we can conclude since  $\|up_n z\|_2^2 = \tau(z^*(p_n y x^* x y^* p_n)z) \leq \tau(z^* z) \leq 1$ .

To get the last statement, consider similarly:

$$\|\alpha\|_2^2 = \|x\|_2^2 = \tau(\alpha u^* u \alpha) = \lim_n \tau(u \alpha p_n \alpha u^*) = \lim_n \tau(p_n \alpha \mathcal{E}(u^* u) \alpha p_n) = \tau(\alpha \mathcal{E}(u^* u) \alpha).$$

The faithfulness of  $\tau$  gives that  $p(\mathcal{E}u^*u)p = p$ . Since  $\mathcal{E}u^*u \leq 1$ , we must also have  $\mathcal{E}(u^*u)p = p\mathcal{E}(u^*u)$  which is enough to conclude.  $\square$

LEMMA 2.3. *Let  $u, v \in L_2(\mathcal{M})$  with  $\|u\|_{\infty,c}, \|v\|_{\infty,c} \leq 1$ , then  $\|\mathcal{E}(u^*v)\|_\infty \leq 1$ .*

*Proof.* As  $\mathcal{E}$  is completely positive, we must have  $\begin{bmatrix} \mathcal{E}(u^*u) & \mathcal{E}(u^*v) \\ \mathcal{E}(v^*u) & \mathcal{E}(v^*v) \end{bmatrix} \geq 0$ . Hence one can find a contraction  $C \in \mathcal{N}$  so that  $\mathcal{E}(u^*v) = \mathcal{E}(u^*u)^{1/2} C \mathcal{E}(v^*v)^{1/2}$  from which the result follows.  $\square$

LEMMA 2.4. *Let  $E$  be a right  $\mathcal{N}$ -submodule of  $L_2(\mathcal{M})$  and  $1 \leq p \leq \infty$ . For any  $x \in E$*

$$\|x\|_{p,c} = \sup_{z \in E, \|z\|_{p',c} \leq 1} |\tau(z^*x)|.$$

*Proof.* Let  $x = u\alpha$  be the polar decomposition given by Lemma 2.2.

First we assume that  $\|x\|_{p,c} < \infty$ , that is  $\alpha \in L_p(\mathcal{N})$ . Let  $z = v\beta$  be the polar decomposition of  $z \in E$  with  $\|z\|_{p',c} \leq 1$ , then  $\beta \in L_{p'}(\mathcal{N})$  has norm less than 1. Let  $p_n = 1_{\{0\} \cup [\frac{1}{n}, n]}$  and  $q_n = 1_{\{0\} \cup [\frac{1}{n}, n]}$ , then as  $x, z \in L_2(\mathcal{M})$

$$\tau(z^*x) = \lim_n \tau(q_n \beta v^* u \alpha p_n) = \lim_n \tau(q_n \beta \mathcal{E}(v^*u) \alpha p_n) = \tau(\beta \mathcal{E}(v^*u) \alpha),$$

thus using Lemma 2.3  $|\tau(z^*x)| \leq \|\alpha\|_p \|\beta\|_{p'} \leq \|\alpha\|_p$ .

To get the reverse inequality even if  $\alpha \notin L_p(\mathcal{N})$ , it suffice to consider  $z_n = up_n \alpha^{p-1} / \|p_n \alpha\|_p^{p-1} \in E$  and let  $n \rightarrow \infty$  if  $1 \leq p < \infty$ . For  $p = \infty$ , one can take  $z_n = ua_n$  where  $a_n \in \mathcal{N}$  is of norm 1 in  $L_1(\mathcal{N})$  with support in  $1_{\alpha > \frac{1}{n}}$  and norming  $\alpha$  at the limit.  $\square$

### 3. Sums of free variables

#### 3.1. Basic facts

A convenient way for us to look at sums of free variables is to see them as elements of length 1 in the free product  $(\mathcal{M}, \tau) = *_{i=1 \dots N} (\mathcal{M}_i, \tau)$ . The trace-preserving conditional expectation from  $\mathcal{M}$  to  $\mathcal{M}_i$  will be denoted by  $\mathcal{E}_i$ .

For  $1 \leq p \leq \infty$ , we denote by  $E_p$  the closure of the span of words of length 1 in  $L_p(\mathcal{M})$ . They are  $\mathcal{N}$ -bimodules and it is well known that the natural orthogonal projection from  $L_p(\mathcal{M})$  to  $E_p$  is bounded and of norm less than 4 (see [23] or [13]).

We recall that we use the notation  $\|x\|_p$  for the  $L_p$ -norm without referring to the underlying algebra as it does not depend on it by the compatibility we impose on traces.

Any  $x \in E_2$  can be decomposed as  $x = \sum_i x_i$  where  $x_i \in \dot{L}_2(\mathcal{M}_i)$  and with  $\|x\|_2^2 = \sum_i \|x_i\|_2^2$ .

The conditioned column norm we introduced in the previous section can be expanded in terms of functions  $x_i$  as  $\mathcal{E}(x^*x) = \sum_i \mathcal{E}(x_i^*x_i)$ .

DEFINITION 3.1. For  $x \in E_2$  and  $p \in [1, \infty]$ , we denote

$$\|x\|_{p,c} = \left\| \left( \sum_i \mathcal{E}(x_i^* x_i) \right)^{1/2} \right\|_p, \quad \|x\|_{p,r} = \left\| \left( \sum_i \mathcal{E}(x_i x_i^*) \right)^{1/2} \right\|_p, \quad \|x\|_{p,d} = \left\| \sum_i x_i \otimes e_i \right\|_{L_p(\mathcal{M} \overline{\otimes} \ell_\infty)},$$

where  $(e_i)$  stands for the standard basis of  $\ell_\infty$ .

Note that  $\|x\|_{p,r} = \|x^*\|_{p,c}$ , that allows to deduce easily results for  $\|\cdot\|_{p,r}$  from those for  $\|\cdot\|_{p,c}$ . Of course,  $\|x\|_{p,d} = (\sum_i \|x_i\|_p^p)^{1/p}$  when  $1 \leq p < \infty$  with the usual modification for  $p = \infty$ .

Viewing  $E_2$  as a subspace of  $L_2(\mathcal{M})$ , we have a notion of  $\mathcal{E}$ -polar decomposition given by Lemma 2.2. More precisely, for any  $x = \sum_i x_i \in E_2$  there is a decomposition  $x = u(\mathcal{E}x^*x)^{1/2}$  where  $u \in E_2$  satisfies  $\|u\|_{\infty,c} \leq 1$  and  $\mathcal{E}x^*x = \sum_i \mathcal{E}x_i^* x_i \in L_1(\mathcal{N})$ . Moreover,  $s(\mathcal{E}x^*x) \leq \mathcal{E}u^*u \leq 1$ .

Lemma 2.4 can also be made more precise in our present context and we obtain:

LEMMA 3.2. Let  $1 \leq p \leq \infty$  and  $x = \sum_i x_i \in E_2$ , then

$$\|x\|_{p,c} = \sup_{z \in E_2, \|z\|_{p',c} \leq 1} |\tau(z^*x)|,$$

where  $\tau(z^*x) = \sum_i \tau(z_i^* x_i)$ .

We recall the Voiculescu inequality which is our fundamental tool. It was first proved in [27] when  $\mathcal{N} = \mathbb{C}$  and with amalgamation in [12]. Any element in  $x \in E_\infty \subset E_2$  can be written as  $x = \sum_i x_i$  where actually  $x_i = \mathcal{E}_i x \in \mathcal{M}_i$  and the sum converges in  $L_2$ . We have the following:

THEOREM 3.3 (Voiculescu). Let  $x = \sum_i x_i \in E_\infty$ , then

$$\max\{\|x\|_{\infty,c}, \|x\|_{\infty,r}, \|x\|_{\infty,d}\} \leq \|x\|_\infty \leq \|x\|_{\infty,c} + \|x\|_{\infty,r} + \|x\|_{\infty,d}.$$

It will be convenient to write for  $x \in E_\infty$ :

$$\|x\|_{\infty,\cap} = \max\{\|x\|_{\infty,c}, \|x\|_{\infty,r}, \|x\|_{\infty,d}\}.$$

Our main goal is to find a version of these inequalities for general fully symmetric spaces. Using duality, one obtains an estimate of the norm of sums of free variables in  $L_1$  given by an infimum.

### 3.2. Algebraic decompositions

The heart of our argument is to obtain an algebraic decomposition in the spirit of [4] where it was done to study Khinchin inequalities. The idea is to look for a decomposition of a sum of free variables in  $L_\infty$  that is optimal for a variant of the dual norm of  $\|\cdot\|_{\infty,\cap}$ . We first justify that such decompositions do exist.

PROPOSITION 3.4. Let  $x \in E_\infty$ , then there exists a decomposition  $x = a + d + b$  with  $a, d, b \in E_2$  such that

$$\inf_{\substack{x = \alpha + \gamma + \beta \\ \alpha, \beta, \gamma \in E_2}} \|\alpha\|_{1,c} + \|\gamma\|_{1,d} + \|\beta\|_{1,r} = \|a\|_{1,c} + \|d\|_{1,d} + \|b\|_{1,r}.$$

Before going into the proof, we prove an intermediate lemma

LEMMA 3.5. Let  $x \in E_\infty$  such that  $x = \alpha + \gamma + \beta$  with  $\alpha, \gamma, \beta \in E_2$ , then there exists another decomposition  $x = a + d + b$  with  $a, d, b \in E_2$  such that  $\|a\|_{1,c} \leq \|\alpha\|_{1,c}$ ,  $\|b\|_{1,r} \leq \|\beta\|_{1,r}$ ,  $\|d\|_{1,d} \leq \|\gamma\|_{1,d}$  and  $\|a\|_{\infty,c}, \|b\|_{\infty,r}, \|d\|_2 \leq 5\|x\|_\infty$ .

*Proof.* Set  $M = \|x\|_\infty$  and define  $e = 1_{[0,M]}((\mathcal{E}\alpha^*\alpha)^{1/2}) \in \mathcal{N}$ ,  $f = 1_{[0,M]}((\mathcal{E}\beta\beta^*)^{1/2}) \in \mathcal{N}$ . We consider the decomposition

$$x = a + d + b = (f\alpha e + x(1-e)) + f\gamma e + (f\beta e + (1-f)xe).$$

Note that  $(f\alpha e)^*(f\alpha e) \leq e(\alpha^*\alpha)e$ . By the operator monotony of the square root

$$(\mathcal{E}(f\alpha e)^*(f\alpha e))^{1/2} \leq (e(\mathcal{E}\alpha^*\alpha)e)^{1/2} = e(\mathcal{E}\alpha^*\alpha)^{1/2}.$$

Since  $\|\cdot\|_{1,c}$  is a norm by Lemma 3.2:

$$\|a\|_{1,c} \leq \|f\alpha e\|_{1,c} + \|x(1-e)\|_{1,c} \leq \tau\left(e(\mathcal{E}\alpha^*\alpha)^{1/2}\right) + \tau((1-e)M) \leq \tau\left((\mathcal{E}\alpha^*\alpha)^{1/2}\right).$$

Similarly,  $\|b\|_{1,r} \leq \|\beta\|_{1,r}$  and  $\|d\|_{1,d} \leq \|\gamma\|_{1,d}$  is obvious.

By the same argument, it is also clear that  $\|a\|_{\infty,c}, \|b\|_{\infty,r} \leq 2M$  (as  $\|x\|_{\infty,c} \leq \|x\|_\infty$ ), hence  $\|a\|_2, \|b\|_2 \leq 2M$ . Thus by the triangle inequality,  $\|d\|_2 \leq 5M$ .  $\square$

*Proof of Proposition 3.4.* Take a sequence of decompositions  $x = a_n + d_n + b_n$  which are optimal up to  $\frac{1}{n}$ . By Lemma 3.5, we can assume that they are uniformly bounded in  $L_2$ . By taking subsequences and convex combinations, we may obtain new sequences  $(a_n)$ ,  $(b_n)$ ,  $(d_n)$  converging in  $L_2$  to  $a, b, d$  with the same properties. We must have  $x = a + d + b$ . As the identity map from  $(E_2, \|\cdot\|_2)$  to  $(E_2, \|\cdot\|_{1,c})$  is continuous (and similarly for  $\|\cdot\|_{1,r}$  and  $\|\cdot\|_{1,d}$ ), we can conclude that  $a, b, d$  achieve the infimum.  $\square$

For  $x \in E_2$ , we set

$$\|x\|_{1,\Sigma} = \inf_{x=a+d+b; a,b,d \in E_2} \|a\|_{1,c} + \|d\|_{1,d} + \|b\|_{1,r}.$$

On  $E_2$ , we have several norms  $\|\cdot\|_{1,\bullet}$  for  $\bullet \in \{c, r, d\}$  as well as  $\|\cdot\|_1$ . The Voiculescu inequality with Lemma 3.2 gives that for  $x \in E_2$ ,  $\|x\|_1 \leq C\|x\|_{1,\bullet}$  for some constant  $C$ .

Assume for now that  $N$  is finite (we add an exponent  $N$  to emphasize it). Since  $\mathring{L}_1(\mathcal{M}_i)$  is 2-complemented in  $L_1(\mathcal{M})$ ,  $\|\cdot\|_1$  and  $\|\cdot\|_{1,d}$  are equivalent on  $E_2^N$ . From all these facts, it follows that  $\|\cdot\|_{1,\Sigma}$  is also a norm on  $E_2^N$  equivalent to  $\|\cdot\|_1$  (with constants possibly depending on  $N$ ). Thus the dual of  $(E_2^N, \|\cdot\|_{1,\Sigma})$  is isomorphic to the dual of  $(E_2^N, \|\cdot\|_1)$  which is  $E_\infty^N$  as a vector space with the usual duality bracket because  $E_1$  is complemented in  $L_1$  by the orthogonal projection and  $E_2$  is dense in  $E_1$  for the  $L_1$ -norm.

Before stating the duality, we need an extra norm on  $E_\infty$  given by, for  $x = \sum_i x_i \in E_\infty$ ,

$$\|x\|_{\infty,\hat{d}} = \sup_i \inf_{e \in \mathcal{N}} \|x_i + e\|_\infty.$$

We clearly have  $\|x\|_{\infty,\hat{d}} \leq \|x\|_{\infty,d} \leq 2\|x\|_{\infty,\hat{d}}$ .

LEMMA 3.6. The dual of  $(E_2, \|\cdot\|_{1,\Sigma})$  is  $E_\infty$  with the norm  $\|\cdot\|_{\infty,\hat{r}} = \max\{\|\cdot\|_{\infty,c}, \|\cdot\|_{\infty,r}, \|\cdot\|_{\infty,\hat{d}}\}$  and anti-duality bracket  $\langle z, x \rangle = \tau(z^*x)$  for  $x \in E_2$  and  $z \in E_\infty$ .

*Proof.* We have already justified the duality as vector spaces assuming  $N$  finite. For the identification of the dual norm, let  $x \in E_\infty^N$ . Using that  $\|\cdot\|_{1,\Sigma}$  corresponds to a sum norm, its dual norm is given by a supremum. By Lemma 3.2, the dual norm on  $E_\infty^N$  of  $(E_2^N, \|\cdot\|_{1,c})$  is exactly  $\|\cdot\|_{\infty,c}$ . The same holds for the row norm and the dual norm on  $E_\infty^N$  of  $(E_2^N, \|\cdot\|_{1,d})$  is clearly  $\|\cdot\|_{\infty,\hat{d}}$ .



Using the Voiculescu inequality once more justifies that  $\|\cdot\|_{1,\Sigma}$  and  $\|\cdot\|_1$  are equivalent on  $E_2^N$  with some universal constant independent of  $N$ .

If we are looking at infinite free products, as  $\cup_{N \in \mathbb{N}^*} E_2^N$  is dense in  $E_2$  for the norm  $\|\cdot\|_2$  (bigger than both  $\|\cdot\|_{1,\Sigma}$  and  $\|\cdot\|_1$ ),  $\|\cdot\|_{1,\Sigma}$  and  $\|\cdot\|_1$  are also equivalent on  $E_2$ . Thus Lemma 3.6 also holds for  $N = \infty$ .  $\square$

We choose this approach to avoid looking at the dual of  $(E_2, \|\cdot\|_{1,c})$  which may be hard to describe.

As in [12], we will need another algebraic construction to make the variable symmetric before finding the algebraic decomposition. We consider the algebras  $\tilde{\mathcal{M}}_i = \mathcal{M}_i \oplus \mathcal{M}_i$  with trace  $\tau((x, y)) = \frac{1}{2}(\tau(x + y))$ . Clearly  $(\mathcal{N}, \tau)$  is identified to a subalgebra of  $(\tilde{\mathcal{M}}_i, \tau)$  by  $n \mapsto (n, n)$ . We simply write  $\mathcal{N} \subset \tilde{\mathcal{M}}_i$  not referring to the inclusion map. As before, the letter  $\tau$  is used for traces on different algebras but this is compatible with our identifications and leads to no confusion.

We consider the free product  $(\tilde{\mathcal{M}}, \tau) = *_{i, \mathcal{N}}(\tilde{\mathcal{M}}_i, \tau)$  with conditional expectation  $\tilde{\mathcal{E}} : \tilde{\mathcal{M}} \rightarrow \mathcal{N}$ . Thus for  $(x, y) \in \tilde{\mathcal{M}}_i$ ,  $\tilde{\mathcal{E}}(x, y) = \frac{1}{2}\mathcal{E}(x + y)$ .

The spaces of words of length one corresponding to  $\tilde{\mathcal{M}}$  in  $L_p$  will be denoted by  $\tilde{E}_p$ .

For  $z_i \in \mathcal{M}_i$ , we write  $\pi(z_i) = (z_i, -z_i) \in \tilde{\mathcal{M}}_i$ . We have that for  $z = \sum_i z_i \in E_\infty$ ,

$$\mathcal{E} \sum_i z_i^* z_i = \tilde{\mathcal{E}} \sum_i \pi(z_i)^* \pi(z_i).$$

In particular using the Voiculescu inequality, this allows to extend  $\pi : (E_\infty, \|\cdot\|_\infty) \rightarrow (\tilde{E}_\infty, \|\cdot\|_\infty)$  by  $\pi(z) = \sum_i \pi(z_i)$  as a bounded map with  $\|z\|_{\infty, \bullet} = \|\pi(z)\|_{\infty, \bullet}$  where  $\bullet \in \{r, c, d\}$ . Similarly one can extend  $\pi : E_2 \rightarrow \tilde{E}_2$  to an isometry for the  $L_2$ -norms.

The swap maps  $\mathcal{S}_i : \tilde{\mathcal{M}}_i \rightarrow \tilde{\mathcal{M}}_i$ ,  $\mathcal{S}_i(x, y) = (y, x)$  are normal trace-preserving  $*$ -representations which are also  $\mathcal{N}$ -bimodular. Thus the free product  $\mathcal{S} = *_{i, \mathcal{N}} \mathcal{S}_i$  extends to an  $*$ -isomorphism  $\mathcal{S} : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}$  which is isometric on all  $L_p(\tilde{\mathcal{M}})$ . Note that  $\mathcal{S}(\pi(x)) = -\pi(x)$  for  $x \in E_\infty$ .

We can conclude about the algebraic decomposition:

**PROPOSITION 3.7.** *Let  $x = \sum_i x_i \in E_\infty$  there exists a sequence  $(u_i)$  with  $u_i \in \mathcal{M}_i$  with*

$$\left\| \left( \mathcal{E} \left( \sum_i u_i^* u_i \right) \right)^{1/2} \right\|_\infty, \quad \left\| \left( \mathcal{E} \left( \sum_i u_i u_i^* \right) \right)^{1/2} \right\|_\infty, \quad \sup_i \|u_i\|_\infty \leq 1$$

and  $\alpha, \beta \in \mathcal{N}^+$  and sequences  $(\gamma_i), (\delta_i)$  in  $\mathcal{M}_i^+$  such that  $x_i = u_i \alpha + \beta u_i + u_i \gamma_i$ ,  $u_i \gamma_i = \delta_i u_i$  and

$$s(\alpha) \leq \mathcal{E}|u|^2 \leq 1, \quad s(\beta) \leq \mathcal{E}|u^*|^2 \leq 1, \quad s(\gamma_i) \leq |u_i|^2 \leq 1, \quad ; \quad s(\delta_i) \leq |u_i^*|^2 \leq 1.$$

**REMARK 3.8.** We point out that  $\sum_i u_i^* u_i$  may only be in  $L_1(\mathcal{M})^+$ . Nevertheless  $\mathcal{E} \sum_i u_i^* u_i$  sits in  $\mathcal{N}$  (similarly for rows).

**REMARK 3.9.** The conditions on  $u_i$  implies that  $s(\gamma_i)$  and  $|u_i|^2$  commute, more precisely  $s(\gamma_i) = s(\gamma_i)|u_i|$ .

*Proof.* Take  $x \in E_\infty$  and consider  $\pi(x) \in \tilde{E}_\infty$ . By Proposition 3.4, we have a decomposition  $\pi(x) = a + b + d$  with  $a, d, b \in \tilde{E}_2$  such that  $\|\pi(x)\|_{1,\Sigma} = \|a\|_{1,c} + \|d\|_{1,d} + \|b\|_{1,r}$ . We may assume that  $\mathcal{S}(a) = -a$  by replacing it with  $a' = \frac{1}{2}(a - \mathcal{S}(a))$  as  $\|a'\|_{1,c} \leq \frac{1}{2}(\|a\|_{1,c} + \|\mathcal{S}(a)\|_{1,c}) = \|a\|_{1,c}$ . The same holds for  $b$  and  $d$ .



Consider the  $\tilde{\mathcal{E}}$ -polar decomposition  $a = v\alpha$  and  $b^* = w^*\beta$  where  $\alpha = (\tilde{\mathcal{E}}a^*a)^{1/2} \in L_1(\mathcal{N})^+$  and  $\beta = (\tilde{\mathcal{E}}bb^*)^{1/2} \in L_1(\mathcal{N})^+$  given by Lemma 2.2. We must also have  $\mathcal{S}(v) = -v$  and  $\mathcal{S}(w) = -w$ . Writing  $d = \sum_i d_i$ , we also consider the usual polar decompositions  $d_i = z_i\gamma_i = \delta_i t_i$  with  $\gamma_i, \delta_i \in L_2(\mathcal{M}_i)^+$  as  $d_i \in L_2(\mathcal{M}_i)$ . We must also have that  $\mathcal{S}(\gamma_i) = \gamma_i$ ,  $\mathcal{S}(z_i) = -z_i$  and  $\mathcal{S}(\delta_i) = \delta_i$ ,  $\mathcal{S}(t_i) = -t_i$ .

By Lemma 3.6 (as  $\tilde{E}_\infty \subset \tilde{E}_2$ ), there is  $u' \in \tilde{E}_\infty$  with  $\|u'\|_{\infty, \tilde{\mathcal{N}}} \leq 1$  so that  $\|\pi(x)\|_{1, \Sigma} = \tau(u'^*\pi(x))$ . As before let  $u = \frac{1}{2}(u' - \mathcal{S}(u')) \in \tilde{E}_\infty$ , we have that  $\mathcal{S}(u) = -u$ ,  $\tau(u^*\pi(x)) = \|\pi(x)\|_{1, \Sigma}$ . We clearly have that  $\|u\|_{\infty, c} \leq \|u'\|_{\infty, c}$  and  $\|u\|_{\infty, r} \leq \|u'\|_{\infty, r}$ . For any  $e \in \mathcal{N}$ ,  $u_i = \frac{1}{2}(u'_i + e - \mathcal{S}(u'_i + e))$ , hence we get that  $\|u\|_{\infty, d} \leq \|u'\|_{\infty, \tilde{d}}$ .

From Proposition 3.4, we infer the equalities

$$\|\alpha\|_1 + \|\beta\|_1 + \sum_i \|\gamma_i\|_1 = \|x\|_{1, \Sigma} = \tau(u^*va + \beta wu^*) + \sum_i \tau(u_i^*z_i\gamma_i).$$

Because of Lemma 2.3,  $\|\tilde{\mathcal{E}}u^*v\|_\infty \leq 1$  and  $\|\tilde{\mathcal{E}}wu^*\|_\infty \leq 1$  and  $\|u_i^*z_i\|_\infty \leq 1$ . Thus, we must have  $\tau((\tilde{\mathcal{E}}u^*v)\alpha) = \|\alpha\|_1$ ,  $\tau((\tilde{\mathcal{E}}wu^*)\beta) = \|\beta\|_1$  and  $\tau(u_i^*z_i\gamma_i) = \|\gamma_i\|_1$  for all  $i$ .

Necessarily with  $p = s(\alpha)$ ,  $p = p(\tilde{\mathcal{E}}u^*v)p = \mathcal{E}((up)^*(vp))$ . In particular,  $\tau((up)^*(vp)) = \tau(p) \geq \|up\|_2 \cdot \|vp\|_2$ . Thus there is equality in the Cauchy-Schwarz inequality so that  $up = vp$  and  $\tau((\tilde{\mathcal{E}}u^*u)p) = \tau(p)$ . But by Lemma 2.2  $a = vp\alpha$  and  $a = up\alpha = u\alpha$ , and  $p \leq \tilde{\mathcal{E}}uu^* \leq 1$ . The same argument also gives that  $b = \beta u$  and  $s(\beta) \leq \tilde{\mathcal{E}}uu^* \leq 1$ .

We also have that  $\tau(u_i^*z_i\gamma_i) = \|\gamma_i\|_1$ . Thus  $s(\gamma_i) = s(\gamma_i)u_i^*z_i$  and as above  $z_i = u_i s(\gamma_i)$  and  $s(\gamma_i) \leq |u_i|^2 \leq 1$ . Similarly  $d_i = \delta_i u_i$  and  $s(\delta_i) \leq |u_i^*|^2 \leq 1$ .

By the Voiculescu inequality,  $\|u\|_\infty \leq 3$  and  $\|\pi(x)\|_\infty \leq 3\|x\|_\infty$ . Note that we have  $\tilde{\mathcal{E}}u^*\pi(x) = \alpha + \mathcal{E}(\sum_i u_i^*\beta u_i + u_i^*u_i\gamma_i) = \alpha + \mathcal{E}(\sum_i u_i^*\beta u_i + \gamma_i)$ . We get  $\|\alpha\|_\infty \leq \|u^*\pi(x)\|_\infty \leq 9\|x\|_\infty$ . The same argument works for  $b$ , thus  $d = x - a - b \in E_\infty$  with  $\|d\|_\infty \leq 19\|x\|_\infty$ . The fact that  $\gamma_i \in \mathcal{M}_i^+$  follows from  $\gamma_i = \tilde{\mathcal{E}}_i u_i^* d_i \in \mathcal{M}_i$ .

To get the desired algebraic decomposition, it suffices to note that as  $\mathcal{S}(u_i) = -u_i \in \tilde{\mathcal{M}}_i$ , it is of the form  $u_i = (r_i, -r_i)$  for some  $r_i \in \mathcal{M}_i$ . The element  $\sum_i r_i^* r_i$  makes sense in  $L_1(\mathcal{M})^+$  as  $\mathcal{E}(r_i^* r_i) = \tilde{\mathcal{E}}(u_i^* u_i)$  and thus  $\tau(r_i^* r_i) = \tau(u_i^* u_i)$ . Similarly  $\gamma_i = (g_i, g_i)$ ,  $\delta_i = (h_i, h_i)$  for some  $g_i, h_i \in \mathcal{M}_i^+$ . Thus we get that  $x_i = r_i \alpha + \beta r_i + r_i g_i = r_i \alpha + \beta r_i + h_i r_i$  and the conclusion on supports follows directly from that of  $u$ .  $\square$

### 3.3. Norm estimates

We will use the algebraic decomposition to get estimates for sums of free variables.

**THEOREM 3.10.** *With the notation of Proposition 3.7, for any fully symmetric function space  $E$*

$$\|x\|_{E(\mathcal{M})} \leq 4\|\alpha\|_{E(\mathcal{N})} + 4\|\beta\|_{E(\mathcal{N})} + 4\left\|\sum_i \gamma_i \otimes e_i\right\|_{E(\mathcal{M} \overline{\otimes} \ell_\infty)},$$

$$\|\alpha\|_{E(\mathcal{N})}, \|\beta\|_{E(\mathcal{N})}, \left\|\sum_i \gamma_i \otimes e_i\right\|_{E(\mathcal{M} \overline{\otimes} \ell_\infty)} \leq 4\|x\|_{E(\mathcal{M})}.$$

*Proof.* We fix  $x = \sum_i x_i \in E_\infty$ .

We start with the upper bound. We have that  $\dot{u} = \sum_i \dot{u}_i$  is a well-defined element of  $E_\infty$  by the Voiculescu inequality as  $0 \leq \sum_{i=1}^N \mathcal{E} \dot{u}_i^* \dot{u}_i \leq \sum_{i=1}^N \mathcal{E} u_i^* u_i$  (similarly for rows) and  $\|\dot{u}_i\|_\infty \leq 2$  for all  $1 \leq i \leq N$ . Thus  $\|\dot{u}\|_\infty \leq 4$ .

We point out that  $\mathcal{E}(x_i) = \mathcal{E}(u_i)\alpha + \beta\mathcal{E}(u_i) + \mathcal{E}(u_i\gamma_i) = 0$ . As  $a = \mathring{u}\alpha$ ,  $b = \beta\mathring{u}$  are well defined,  $d = x - a - b$  also is and necessarily  $\mathcal{E}_i d = (u_i\gamma_i)$ , so we can write  $x = \mathring{u}\alpha + \beta\mathring{u} + \sum_i (u_i\gamma_i)$ .

It is clear that  $\|\mathring{u}\alpha\|_{E(\mathcal{M})} \leq 4\|\alpha\|_{E(\mathcal{N})}$  and similarly  $\|\beta\mathring{u}\|_{E(\mathcal{M})} \leq 4\|\beta\|_{E(\mathcal{N})}$ .

Next we prove that  $\|\sum_i (u_i\gamma_i)\|_{E(\mathcal{M})} \leq 4\|\sum_i \gamma_i \otimes e_i\|_{E(\mathcal{M} \overline{\otimes} \ell_\infty)}$ . Let  $t > 0$ , we recall that for any variable in a non-commutative measure space  $\int_0^t \mu(z) = \|z\|_{L_1+tL_\infty}$ . Consider an optimal decomposition  $\sum_i \gamma_i \otimes e_i = r + s$  in  $(L_1+tL_\infty)(\oplus \mathcal{M}_i)$ . We have  $\gamma_i = r_i + s_i$  and we may assume that  $0 \leq r_i, s_i \leq \gamma_i$ .

We have  $\sum_i (u_i\gamma_i) = \sum_i (u_i r_i) + \sum_i (u_i s_i)$  and  $\|r\|_1 = \sum_i \|r_i\|_1$ ,  $\|s\|_\infty = \sup_i \|s_i\|_\infty$ .

By the Voiculescu inequality, we can control  $\|\sum_i (u_i s_i)\|_\infty$ . Indeed  $\sup_i \|(u_i s_i)\|_\infty \leq 2 \sup_i \|s_i\|_\infty$ . And

$$\left\| \sum_i (u_i s_i) \right\|_{\infty, c}^2 \leq \left\| \sum_i \mathcal{E}(s_i u_i^* u_i s_i) \right\|_\infty.$$

But  $s_i u_i^* = u_i^* u_i s_i u_i^*$  as  $s(\gamma_i) \leq s(\gamma_i) \leq u_i^* u_i \leq 1$ . Thus  $0 \leq \mathcal{E} s_i u_i^* u_i s_i \leq \sup_k \{\|s_k\|_\infty^2\} \mathcal{E} u_i^* u_i$  and it follows that  $\|\sum_i (u_i s_i)\|_{\infty, c} \leq \sup_k \{\|s_k\|_\infty\} \|u\|_{\infty, c} \leq \sup_k \{\|s_k\|_\infty\}$ .

The last term is easier to handle

$$\left\| \sum_i (u_i s_i) \right\|_{\infty, r} \leq \left\| \sum_i \mathcal{E}(u_i s_i s_i u_i^*) \right\|_\infty^{1/2} \leq \sup_k \{\|s_k\|_\infty\} \|u\|_{\infty, r}.$$

We obtain  $\|\sum_i (u_i s_i)\|_\infty \leq 4 \sup_k \{\|s_k\|_\infty\}$ .

By the triangle inequality  $\|\sum_i (u_i r_i)\|_1 \leq 2 \sum_i \|r_i\|_1$ . Thus we get  $\|\sum_i (u_i \gamma_i)\|_{L_1+tL_\infty} \leq 4\|\sum_i \gamma_i \otimes e_i\|_{L_1+tL_\infty}$  and the estimate we claimed follows.

We turn to the lower bound. First, we note that  $\mathcal{E}(\mathring{u}^* x) = \alpha + \sum_i \mathcal{E}(u_i^* \beta u_i + \gamma_i)$ . Indeed for any  $j$ , we have  $\mathcal{E} \mathring{u}_j^* x = \mathcal{E} \mathring{u}_j^* x_j = \mathcal{E} u_j^* x_j$  because  $x = \sum_i x_i$  is centered. Thus for any  $1 \leq K \leq N$   $\mathcal{E}((\sum_{i=1}^K \mathring{u}_i)x) = \mathcal{E}(\sum_{i=1}^K u_i^* u_i \alpha + u_i^* \beta u_i + \gamma_i)$ . We conclude by letting  $K \rightarrow \infty$  and taking limits in  $L_2$ .

We get  $\|\mathcal{E}(\mathring{u}^* x)\|_{E(\mathcal{N})} \geq \|\alpha\|_{E(\mathcal{N})}$ . Since  $\mathcal{E}$  is always a contraction,  $\|\alpha\|_{E(\mathcal{N})} \leq \|\mathring{u}^* x\|_{E(\mathcal{M})} \leq 4\|x\|_{E(\mathcal{M})}$ . Similarly  $\mathcal{E}(x \mathring{u}^*) = \beta + \mathcal{E}(\sum_i u_i \alpha_i u_i^* + \delta_i)$  gives  $\|\beta\|_{E(\mathcal{N})} \leq 4\|x\|_{E(\mathcal{M})}$ .

Fix  $t > 0$  and let  $p = \oplus p_i \in \oplus_i \mathcal{M}_i$  be such that  $\|p\|_1 = t$ ,  $\|p\|_\infty = 1$  and  $\sum_i \tau(p_i \gamma_i) = \|\sum_i \gamma_i \otimes e_i\|_{L_1+tL_\infty}$ . We can also assume that  $0 \leq p_i \leq s(\gamma_i)$  by replacing it by  $s(\gamma_i)(\frac{p_i + p_i^*}{2}) + s(\gamma_i)$ . Thus by Remark 3.9,  $p_i |u_i| = p_i = |u_i| p_i$ . We use that  $\gamma_i = u_i^* x_i - |u_i|^2 \alpha - u_i^* \beta u_i$ . We have  $\sum_i \tau(p_i |u_i|^2 \alpha) = \sum_i \tau(|u_i| p_i |u_i| \alpha) \geq 0$  and also  $\sum_i \tau(u_i p_i u_i^* \beta) \geq 0$ .

Finally, as  $x_i$  is centered,  $\tau(p_i u_i^* x_i) = \tau((u_i p_i)^* x)$ . Let  $\varepsilon_i$  be the argument of  $\tau(p_i u_i^* x_i)$ . We can check that  $\|\sum_i \varepsilon_i (u_i p_i)\|_{\infty, c} \leq \|\sum_i \varepsilon_i u_i p_i\|_{\infty, c} \leq 1$ ; indeed again

$$\left\| \sum_i \varepsilon_i u_i p_i \right\|_{\infty, c} = \left\| (\mathcal{E} \sum_i |u_i| p_i |u_i|)^{1/2} \right\|_\infty \leq 1.$$

In the same way,  $\|\sum_i (u_i p_i)\|_{\infty, r} \leq 1$  and  $\sup_i \|(u_i p_i)\|_\infty \leq 2$ . The element  $X = \sum_i \varepsilon_i (u_i p_i)$  is well defined in  $\mathcal{M}$  with norm less than 4 due to the Voiculescu inequality. By the triangle inequality,  $\|X\|_1 \leq 2 \sum_i \|p_i\|_1 = 2t$ . Thus gathering the inequalities

$$\left\| \sum_i \gamma_i \otimes e_i \right\|_{L_1+tL_\infty} \leq \sum_i |\tau(p_i u_i^* x_i)| = \tau \left( X^* \cdot \left( \sum_i x_i \right) \right) \leq 4\|x\|_{L_1+tL_\infty}.$$

As  $E$  is fully symmetric,  $\|\sum_i \gamma_i \otimes e_i\|_{E(\mathcal{M} \overline{\otimes} \ell_\infty)} \leq 4\|x\|_{E(\mathcal{M})}$ .  $\square$

For a fully symmetric space  $E$ , define for  $x \in E_\infty$

$$\|x\|_{E,\Sigma} = \inf_{a,b,d \in E_\infty; x=a+d+b} \|(\mathcal{E}a^*a)^{1/2}\|_{E(\mathcal{N})} + \|(\mathcal{E}bb^*)^{1/2}\|_{E(\mathcal{N})} + \left\| \sum_i d_i \otimes e_i \right\|_{E(\mathcal{M} \overline{\otimes} \ell_\infty)},$$

as well as

$$\|x\|_{E,\cap} = \max \left\{ \|(\mathcal{E}x^*x)^{1/2}\|_{E(\mathcal{N})}, \|(\mathcal{E}xx^*)^{1/2}\|_{E(\mathcal{N})}, \left\| \sum_i x_i \otimes e_i \right\|_{E(\mathcal{M} \overline{\otimes} \ell_\infty)} \right\}.$$

To simplify we also write  $\|x\|_{E,c} = \|(\mathcal{E}x^*x)^{1/2}\|_{E(\mathcal{N})}$  and similarly for  $r$  and  $d$ .

We can have another estimate on the algebraic decomposition.

**PROPOSITION 3.11.** *With the notation of Proposition 3.7, for any fully symmetric function space  $E$ , we have*

$$\|\alpha\|_{E(\mathcal{N})} \leq \|x\|_{E,c}, \quad \|\beta\|_{E(\mathcal{N})} \leq \|x\|_{E,r}, \quad \left\| \sum_i \gamma_i \otimes e_i \right\|_{E(\mathcal{M} \overline{\otimes} \ell_\infty)} \leq \|x\|_{E,d}.$$

*Proof.* We also keep the notation of the proof of Theorem 3.10. This is just a variation.

We have already explained that  $\mathcal{E}(\hat{u}^*x) \geq \alpha \geq 0$ . By Lemma 2.3, there is a contraction  $C \in \mathcal{N}$  so that  $\mathcal{E}(\hat{u}^*x) = (\mathcal{E}(\hat{u}^*\hat{u}))^{1/2}C(\mathcal{E}(x^*x))^{1/2}$  and we can get  $\|\alpha\|_{E(\mathcal{N})} \leq \|x\|_{E,c}$ . The same works for  $\beta$ .

For the last bound, we modify the arguments from Theorem 3.10. With the same notation, one just need to note that  $\sum_i |\tau(p_i u_i^* x_i)| = \tau((\sum_i \varepsilon_i p_i u_i \otimes e_i) \cdot (\sum_i x_i \otimes e_i))$ . The element  $g = \sum_i \varepsilon_i p_i u_i \otimes e_i$  also satisfies  $\|g\|_\infty \leq 1$ ,  $\|g\|_1 \leq t$  and we get  $\|\sum_i \gamma_i \otimes e_i\|_{L_1+tL_\infty} \leq \|\sum_i x_i \otimes e_i\|_{L_1+tL_\infty}$ . This yields the result for all fully symmetric spaces.  $\square$

We can deduce the Rosenthal type inequalities in the spirit of [6, 13, 14].

**COROLLARY 3.12.** *For any fully symmetric space  $E$  and  $x \in E_\infty$ , we have*

$$\frac{1}{16} \|x\|_{E,\Sigma} \leq \|x\|_{E(\mathcal{M})} \leq 12 \|x\|_{E,\cap}.$$

*If moreover  $E$  is an  $(L_1, L_2)$ -interpolation space,  $\|x\|_{E(\mathcal{M})} \leq 2 \|x\|_{E,\Sigma}$ .*

*If moreover  $E$  is an  $(L_2, L_\infty)$ -interpolation space,  $\|x\|_{E,\cap} \leq 2 \|x\|_{E(\mathcal{M})}$ .*

*Proof.* One can take  $a = \sum_i \hat{u}_i \alpha$ ,  $b = \sum_i \beta \hat{u}_i$  and  $d = \sum_i u_i \hat{\gamma}_i$ . We have already shown that  $a, b, d \in E_\infty$  and we can use Theorem 3.10 to get the lower bound as  $\|a\|_{E,c} \leq \|\alpha\|_{E(\mathcal{N})}$ ,  $\|b\|_{E,r} \leq \|\beta\|_{E(\mathcal{N})}$  and  $\|d\|_{E,d} \leq 2 \|\sum_i \gamma_i \otimes e_i\|_{E(\mathcal{M} \overline{\otimes} \ell_\infty)}$ . The upper bound is direct from Propositions 3.10 and 3.11 using the triangle inequality.

We justify the remaining inequality when  $E$  is an  $(L_1, L_2)$ -interpolation space.

Let  $a \in E_\infty$  with  $\mathcal{E}$ -polar decomposition  $a = u\alpha$  and  $p = s(\alpha) \in \mathcal{N}^+$ . The map  $r \mapsto ur$  defined on  $p\mathcal{N}p$  extends to a contraction  $L_2(p\mathcal{N}p) \rightarrow L_2(\mathcal{M})$  and also to  $L_1(p\mathcal{N}p) \rightarrow L_1(\mathcal{M})$  as  $\mathcal{E}(ru^*ur)^{1/2} \leq r$  for  $r \in p\mathcal{N}^+p$ . Thus by interpolation  $\|u\alpha\|_{E(\mathcal{M})} \leq \|\alpha\|_{E(\mathcal{N})} = \|(\mathcal{E}a^*a)^{1/2}\|_{E(\mathcal{N})}$ . The same works for  $b$ . For the other term, we proceed similarly by considering the map defined on  $(L_1 + L_2)(\oplus \mathcal{M}_i)$  by  $T(\sum_i d_i \otimes e_i) = \sum_i \hat{d}_i$ .  $T$  has norm smaller than 2 on all interpolation spaces between  $L_1$  and  $L_2$ .

When  $E$  is an  $(L_\infty, L_2)$ -interpolation space, the inequality follows in the same way. Indeed by interpolation, for  $x \in E_\infty$   $\|\mathcal{E}(x^*x)\|_{L_1+tL_\infty} \leq \|x^*x\|_{L_1+tL_\infty}$  which implies  $\|x\|_{E,c} \leq \|x\|_{E(\mathcal{M})}$  by [16]; similarly for rows. The map  $z \in \mathcal{M} \mapsto \sum_i (\mathcal{E}_i(z) - \mathcal{E}(z)) \otimes e_i$  has norm less than 2 on  $L_\infty$  and  $L_2$ .  $\square$

REMARK 3.13. Actually the inequality  $\|x\|_{E(\mathcal{M})} \geq c\|x\|_{E,\Sigma}$  implies  $\|x\|_{E(\mathcal{M})} \leq 12c\|x\|_{E,\cap}$  for all  $E$  and  $x \in E_\infty$  by duality. Indeed, take  $y \in L_\infty \cap tL_1(\mathcal{M})$  norming  $x$  in  $L_1 + tL_\infty$  and apply the inequality to  $z = P_1(y)$ , its component of length one, for  $E = L_\infty \cap tL_1$ . We get a decomposition  $z = a + b + d$  which gives by duality that  $\|x\|_{L_1+tL_\infty} \leq 4c\|x\|_{L_1+tL_\infty,\cap}$ . This allows to conclude to  $\|x\|_{E(\mathcal{M})} \leq 12c\|x\|_{E,\cap}$  for all  $E$ .

REMARK 3.14. Free or independent variables are examples of martingales. The above corollary can be deduced from the literature when  $E$  is an interpolation space between  $L_p$  and  $L_q$  but only when  $2 < p < q < \infty$  in [5, 6] or  $1 < p < q < 2$  in [21] under the Fatou assumption for  $E$  (the results are given in terms of Boyd indices which is slightly stronger). A related estimate is obtained when  $2 = p < q = 4$  in [9] but for a somehow different norms on the right, see Remark 4.7. The novelty is that we allow the end points of the scales, the constants are universal and the decomposition is independent of  $E$ .

#### 4. Applications to other inequalities

From now on, to lighten the notation, when no confusion can occur, we may simply write  $\|x\|_E$  for  $\|x\|_{E(\mathcal{M})}$  if  $x \in L_0(\mathcal{M}, \tau)$ . We do not lose information this way since  $\mu(x)$  does not depend on the semi-finite von Neumann subalgebra in which it is computed as long as the traces are compatible.

##### 4.1. Links with the Johnson–Schechtman inequalities

The Johnson–Schechtman inequality [10] is a very efficient tool to compute explicitly the norm of sums of independent (commutative) variables. Its free analogue was considered in [25]. It concerns only variables with trivial amalgamation. Here we explain how to recover it from our arguments as an algebraic decomposition can be given explicitly. As above  $\mathcal{M}$  is the free product of non-commutative probability spaces  $(\mathcal{M}_i, \tau)$ ,  $1 \leq i \leq N$ , with  $\mathcal{N} = \mathbb{C}$ . This will yield much nicer constants than the original proof. We will concentrate only on the case of symmetric variables as in [25], that is, self-adjoint variables  $x$  such that  $x$  and  $-x$  are equimeasurable.

We give ourselves a fully symmetric function space  $E$  on  $(0, \infty)$  with  $\|1_{[0,1]}\|_E = 1$ . With this normalization, we always have  $\|f\|_1 \leq \|f\|_E \leq \|f\|_\infty$  for any  $f \in L_\infty$  supported on a measure 1 set. Define as in [25]

$$\|f\|_{Z_E^2} = \|\mu(f)1_{[0,1]}\|_E + \|\min\{\mu(f)(1), \mu(f)\}\|_2.$$

PROPOSITION 4.1. *Let  $x_i$  be symmetrically distributed variables in  $\mathcal{M}_i$ , then*

$$\frac{1}{3} \left\| \sum_i x_i \otimes e_i \right\|_{Z_E^2} \leq \left\| \sum_i x_i \right\|_E \leq 3 \left\| \sum_i x_i \otimes e_i \right\|_{Z_E^2}.$$

*Proof.* By enlarging the algebras and using compositions with complete isometries, we can assume that  $\mathcal{M}_i = L_\infty[-\frac{1}{2}, \frac{1}{2}]$  and the  $x_i$  are odd functions.

Assume  $f = \sum_i x_i \otimes e_i \in L_1 + L_\infty(\mathcal{M} \bar{\otimes} \ell_\infty)$  with  $\|f\|_{Z_E^2} < \infty$ . Set  $t = \mu(f)(1)$  and  $\alpha = \|\min\{t, \mu(f)\}\|_2$ , note that  $\alpha \geq t$ .

We can find even projections  $q_i$  in  $\mathcal{M}_i$  so that  $1_{|x_i|>t} \leq q_i \leq 1_{|x_i| \geq t}$  and  $\sum_i q_i |x_i| \otimes e_i$  has distribution  $\mu(f)1_{[0,1]}$ .

Considering the polar decomposition  $x_i = r_i |x_i|$ , we define  $v_i = r_i q_i$ ,  $w_i = \frac{1}{t} x_i (1 - q_i)$ . They are disjointly supported. Let  $u_i = v_i + w_i$ . Because  $x_i$  is an odd function,  $v_i$ ,  $w_i$  and  $u_i$  also are. Clearly  $\|u_i\|_\infty \leq 1$  and  $|\sum u_i \otimes e_i|$  has the same distribution as  $\min\{1, |f|/t\}$ . Hence we have that  $\|\sum_i u_i\|_2 = \alpha/t$ . By the Voiculescu inequality,  $\|\sum_i u_i\|_\infty \leq 2\|\sum_i u_i\|_2 + 1 \leq 3\alpha/t$ .

Similarly by construction the support of  $\sum_i v_i \otimes e_i$  has measure 1, from which we deduce that  $\|\sum_i v_i\|_\infty \leq 3$ .

We have a decomposition  $x_i = tu_i + v_i(|x_i| - t)_+$ . We set  $\gamma_i = (|x_i| - t)_+ q_i$ .

We have a first trivial estimate  $\|\sum_i tu_i\|_E \leq \|\sum_i tu_i\|_\infty \leq 3\alpha$ .

By construction  $\|\sum_i \gamma_i \otimes e_i\|_E \leq \|\mu(f)1_{[0,1]}\|_E$ . Using the same argument as in Theorem 3.10, we get  $\|\sum_i v_i \gamma_i\|_E \leq 3\|\sum_i \gamma_i \otimes e_i\|_E$  (there is no centering).

Thus we arrive at  $\|\sum_i x_i\|_E \leq 3\|\sum_i x_i \otimes e_i\|_{Z_E^2}$ .

In the opposite direction, assume  $\sum x_i \in E$ . Since,

$$3\alpha/t \left\| \sum_i x_i \right\|_1 \geq \tau \left( \left( \sum_i u_i \right)^* \left( \sum_i x_i \right) \right) \geq t \left\| \sum_i u_i \right\|_2^2 = \alpha^2/t,$$

we obtain  $\|\sum_i x_i\|_E \geq \|\sum_i x_i\|_1 \geq \alpha/3$ .

Taking  $\sum_i p_i \otimes e_i$  with functions  $p_i$  even norming  $\sum_i |x_i| q_i$  in  $L_1 + tL_\infty$  and arguing as in Theorem 3.10, one gets  $\|\sum_i |x_i| q_i \otimes e_i\|_E \leq 3\|\sum_i x_i\|_E$  so that we get  $\|\mu(f)1_{[0,1]}\|_E \leq 3\|\sum_i x_i\|_E$ .  $\square$

#### 4.2. Application to martingales

Sums of free variables are basic examples of martingales. In this short section, we explain how the norm estimates we got can be used to interpolate the Burkholder inequality for non-commutative martingales quite easily. The basic idea is to realize the norm in the Burkholder–Gundy inequality as the norm of a sum of free variables with amalgamation.

Let  $(\mathcal{N}_k)_{k=0,\dots,d}$  be a finite filtration of the finite von Neumann algebra  $(\mathcal{N}, \tau)$  ( $\mathcal{N}_d = \mathcal{N}$ ) with conditional expectations  $\mathbb{E}_k$ .

As usual for an element  $x \in \mathcal{N}$ , we consider its martingale difference  $(dx_k)_{k=0}^d$ , where  $dx_0 = \mathbb{E}_0 x$  and  $dx_k = \mathbb{E}_k x - \mathbb{E}_{k-1} x$  for  $k \geq 1$ .

Denote  $\mathcal{M}_k = \mathcal{N} *_{\mathcal{N}_k} \mathcal{N}_{k+1}$  for  $k = 0, \dots, d-1$ . The copy of  $\mathcal{N}$  in  $\mathcal{M}_k$  will be still denoted by  $\mathcal{N}$ , and we denote by  $\rho_k : \mathcal{N}_{k+1} \rightarrow \mathcal{M}_k$  the natural inclusion. The conditional expectation  $\mathcal{M}_k \rightarrow \mathcal{N}_k$  onto the amalgam is denoted by  $\mathcal{E}_k$ , whereas  $\mathcal{E}$  will denote the conditional expectation onto the copy of  $\mathcal{N}$ . Thus  $\mathcal{E}_k = \mathcal{E} \mathcal{E}_k = \mathcal{E}_k \mathcal{E} = \mathbb{E}_k \mathcal{E}$ .

We have that for any  $x \in \mathcal{N}_{k+1}$ ,  $\mathcal{E} \rho_k(x) = \mathcal{E}_k \rho_k(x) = \mathcal{E}_k x = \mathbb{E}_k x$ .

Let  $(\mathcal{M}, \mathcal{E}) = *_{\mathcal{N}}(\mathcal{M}_k, \mathcal{E})$  and denote by  $\mathcal{F}_k, \mathcal{F}_k^+$  the conditional expectation from  $\mathcal{M}$  to  $\rho_k(\mathcal{N}_k) = \mathcal{N}_k \subset \mathcal{M}_k$  and to  $\rho_k(\mathcal{N}_{k+1}) \subset \mathcal{M}_k$  for  $k \geq 0$ . Hence, for  $x_k \in \mathcal{M}_k$ , we have  $\mathcal{F}_k(x_k) = \mathcal{E}_k(x_k)$  and  $\mathcal{E} \mathcal{F}_k^+(x_k) = \mathcal{E}_k(x_k)$ .

Define  $\gamma : \mathcal{N} \rightarrow \mathcal{M}$  by  $\gamma(x) = \mathbb{E}_0 x + \sum_{k=1}^d \rho_{k-1}(dx_k)$ . And let  $\varphi : \mathcal{M} \rightarrow \gamma(\mathcal{N})$  given by  $\varphi(x) = \mathcal{F}_0 x + \sum_{k=0}^{d-1} (\mathcal{F}_k^+ - \mathcal{F}_k)(x)$ . Note that  $(\mathcal{F}_k^+ - \mathcal{F}_k)(x)$  is centered in  $\mathcal{M}_k$  with respect to  $\mathcal{E}$  and  $\varphi(x)$  is a sum of centered free variables up to  $\mathbb{E}_0 x$ .

**PROPOSITION 4.2.** *The map  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$  is a projection onto  $\gamma(\mathcal{N})$  and it extends to a bounded map  $L_p(\mathcal{M}) \rightarrow L_p(\mathcal{M})$  for  $1 < p < \infty$  (with a constant independent of  $d$ ).*

We recall the dual Doob inequality from [11].

**THEOREM 4.3.** *Let  $1 \leq p < \infty$ , there is a constant  $c_p$  (only depending on  $p$ ) such that for any  $d \geq 1$  and  $a_k \in \mathcal{N}^+$ :*

$$\left\| \sum_{k=0}^{d-1} \mathbb{E}_k a_k \right\|_p \leq c_p \left\| \sum_{k=0}^{d-1} a_k \right\|_p.$$

*Proof of Proposition 4.2.* The first point is clear by construction.

To see that  $\varphi$  is bounded on  $L_p(\mathcal{M})$ , we rely on the dual Doob inequality above. Assume first that  $p \geq 2$ . The formal projection  $P_1 : L_p(\mathcal{M}) \rightarrow E_p \subset L_p(\mathcal{M})$  onto words of length 1 has norm less than 4 (see [23]). Note that for  $x \in \mathcal{M}$  if  $P_1(x) = \sum_{k=0}^{d-1} x_k$ , then  $\varphi(x) = \mathbb{E}_0 \mathcal{E}x + \sum_{k=0}^{d-1} \mathcal{F}_k^+(x_k)$ . Thus, define a map on  $E_\infty$  by  $T(\sum_{k=0}^{d-1} x_k) = \sum_{k=0}^{d-1} \mathcal{F}_k^+(x_k)$  so that  $\varphi(x) = T(P_1(x)) + \mathbb{E}_0 \mathcal{E}x$  for  $x \in \mathcal{M}$ . Hence we need to justify that  $T$  is bounded for the  $L_p$ -norm independently of  $d$ . It suffices to check it for the three norms appearing in the free Rosenthal inequality. For the column norm, first note that

$$\mathcal{E}|\mathcal{F}_k^+(x_k)|^2 \leq \mathcal{E}\mathcal{F}_k^+|x_k|^2 = \mathcal{E}_k|x_k|^2 = \mathbb{E}_k\mathcal{E}|x_k|^2.$$

From the dual Doob inequality in  $L_{p/2}$  with  $a_k = \mathcal{E}|x_k|^2$ , we get that  $T$  is bounded for the column norm.

The row estimate is similar, whereas the diagonal one is easy (with no constant depending on  $p$ ).

It is algebraically clear that  $\varphi$  is the projection onto  $\gamma(\mathcal{N})$ . Noticing then that  $\varphi$  is an orthogonal projection on  $L_2$  also gives that it extends to a bounded map on  $L_{p'}$  with the same norm.  $\square$

Using our main estimate, we get that the norms on  $\mathcal{N}$  given  $\|\gamma(x)\|_{E(\mathcal{M})}$  are compatible with interpolation.

**COROLLARY 4.4.** *If  $E$  is an interpolation space for  $(L_2, L_p)$  with  $2 < p < \infty$ , then for  $x \in \mathcal{N}$*

$$\begin{aligned} \|x\|_E \approx_p & \left\| \left( |\mathbb{E}_0 x|^2 + \sum_{k=1}^d \mathbb{E}_{k-1}(|dx_k|^2) \right)^{1/2} \right\|_E + \left\| \left( |\mathbb{E}_0 x^*|^2 + \sum_{k=1}^d \mathbb{E}_{k-1}(|dx_k^*|^2) \right)^{1/2} \right\|_E \\ & + \left\| \sum_{k=0}^d dx_k \otimes e_k \right\|_{E(\mathcal{N} \overline{\otimes} \ell_\infty)} \end{aligned}$$

*If  $E$  is an interpolation space for  $(L_p, L_2)$  with  $1 < p \leq 2$ , then*

$$\begin{aligned} \|x\|_E \approx_p \inf_{x=a+b+c; a,b,c \in \mathcal{N}} & \left\{ \left\| \left( |\mathbb{E}_0 a|^2 + \sum_{k=1}^d \mathbb{E}_{k-1}(|da_k|^2) \right)^{1/2} \right\|_E \right. \\ & \left. + \left\| \left( |\mathbb{E}_0 b^*|^2 + \sum_{k=1}^d \mathbb{E}_{k-1}(|db_k^*|^2) \right)^{1/2} \right\|_E + \left\| \sum_{k=0}^d dc_k \otimes e_k \right\|_{E(\mathcal{N} \overline{\otimes} \ell_\infty)} \right\}. \end{aligned}$$

*Moreover,  $a, b, c$  can be chosen to be independent of  $E$  and  $p$ .*

*Proof.* For  $\infty > p \geq 2$  by corollary 3.12, the quantity on the right-hand side is equivalent to  $\|\gamma(x)\|_E$ . By the Burkholder inequality [14],  $\|\gamma(x)\|_p \approx_p \|x\|_p$ . Thus the map  $\gamma : \mathcal{N} \rightarrow \mathcal{M}$  and  $\gamma^{-1}\varphi : \mathcal{M} \rightarrow \mathcal{N}$  extend to bounded maps on  $E$  by interpolation.

For  $1 < p \leq 2$ , the argument is similar, we also have  $\|x\|_E \approx_p \|\gamma(x)\|_E$  by the Burkholder inequality. Then one has to check that the quantity on the right-hand side is also equivalent to  $\|\gamma(x)\|_E$ . Clearly we can assume  $\mathbb{E}_0 x = 0$  by changing the constants. Then, if  $\gamma(x) = a' + b' + c'$  with  $a', b', c' \in E_\infty$  given by Corollary 3.12, then  $\gamma(x) = \varphi(a') + \varphi(b') + \varphi(c')$ , and  $\varphi(a') = \gamma(a)$  for some  $a \in \mathcal{N}$  with  $\mathbb{E}_0 a = 0$  and  $\|(\sum_{k=1}^d \mathbb{E}_{k-1}(|da_k|^2))^{1/2}\|_E = \|T(a')\|_{E,c}$ . We need to justify that  $T$  is bounded for  $\|\cdot\|_{E,c}$ . First we can assume  $\mathcal{N}_*$  is separable because of deal with finite families. By Proposition 2.8 in [11], there are a von Neumann algebra  $\tilde{\mathcal{M}}$ , maps  $u_q : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$  for  $1 < q < \infty$ , that are compatible in the sense of interpolation such that  $\|u_q(x)\|_q = \|x\|_{q,c}$ ;

moreover the closure of the range of  $u_q$  is complemented in  $L_q(\tilde{\mathcal{M}})$ . Thus we obtain by interpolation that  $\|T(a')\|_{E,c} \leq c_p \|a'\|_{E,c}$  as we have shown in Proposition 4.2 that  $TP_1$  is bounded for the norm  $\|\cdot\|_{p,c}$  and  $\|\cdot\|_{2,c}$ . The argument for  $b'$  is similar and simpler for  $c'$ .  $\square$

REMARK 4.5. Instead of using Proposition 2.8 in [11], we could have used our main estimate. Indeed, one can show for instance that if  $(\mathcal{N}, \tau) \subset (\mathcal{M}, \tau)$  and a map  $T$  on  $\mathcal{M}$  is bounded for the norms  $\|\cdot\|_{p_i}$  and  $\|\cdot\|_{p_i,c}$  with  $1 \leq p_0, p_1 \leq 2$ , then  $T$  is bounded for the norm  $\|\cdot\|_{E,c}$  for any  $(L_{p_0}, L_{p_1})$ -interpolation space. This follows from a standard limit procedure that can be found in [12]. Let  $\tilde{\mathcal{M}}_n$  be the free product over  $M_n(\mathcal{N})$  of  $n$  copies of  $M_n(\mathcal{M})$  with amalgamation over  $M_n(\mathcal{N})$ . For  $x \in \mathcal{M}$ , denote by  $\pi_i(x)$  its  $i$ th copy. One can check using Corollary 3.12 that with  $\gamma_n(x) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \pi_j(x) \otimes e_{j,1}$ , we have  $\lim_n \|\gamma_n(x)\|_E \approx \|x\|_{E,c}$ . Moreover, the closure of the range of  $\gamma_n$  is complemented in  $\tilde{\mathcal{M}}_n$  (uniformly in  $n$ ) in any fully symmetric space. This is enough to conclude, we leave the details to the interested reader.

REMARK 4.6. As we already pointed it out in Remark 3.14, the result was known from [5, 21] in terms of Boyd indices but one cannot use  $L_2$  as an end point without extra assumptions on  $E$ . We also recover as in [21] that when  $p < 2$ , the decomposition in the infimum can be chosen independently of  $E$ . Looking carefully at the proof shows that when  $p$  goes to 1 or  $\infty$ , the equivalence constants are not optimal when  $E = L_p$ . It is known from [8] that one can actually find a decomposition independent of  $p$  for  $E = L_p$ ,  $1 < p \leq 2$  with an optimal behavior of the constants.

REMARK 4.7. If we assume merely that  $E$  is a fully symmetric function space on  $(0,1)$  (with  $\|1_{(0,1)}\|_E = 1$ ), it is possible to somehow extend  $E$  to fully symmetric space on  $(0, \infty)$ . One can choose, for instance, for  $g \in L_0(0, \infty)$ ,  $\|g\|_F = \|\mu(g)1_{[0,1]}\|_E$  or  $\|g\|_{Z_E^2} = \|g\|_F + \|g\|_{L_1+L_2}$ . It is equivalent to the definition of  $Z_E^2$  above and moreover for  $f$  supported on  $(0,1]$ , then  $\|f\|_E \approx \|f\|_{Z_E^2}$ . With any of these constructions, Corollary 4.4 implies that the Johnson–Schechtman inequality in Theorem 1.5 in [9] is true if  $E$  is an interpolation space for  $(L_2, L_p)$  for some  $2 \leq p < \infty$ . It was stated only for  $p = 4$  there. Actually  $Z_E^2$  is up to some constant an interpolation space for  $(L_2, L_p)$  if  $E$  is and it is somehow the bigger norm on  $(0, \infty)$  extending  $E$  with that property.

#### 4.3. Other Rosenthal inequalities

There are many places where the norms  $\|\cdot\|_{E,\cap}$  or  $\|\cdot\|_{E,\Sigma}$  appear. Just as in the previous section, it is possible to interpolate norm inequalities knowing the result for  $L_p$ . Corollary 3.18 in [17] is one example. We simply state the result leaving the proof to the interested reader. For an element in the free product  $(\mathcal{M}, \tau) = *_{\mathcal{N}}(\mathcal{M}_i, \tau)$ , we write  $x \in \mathcal{L}_i \cap \mathcal{R}_i$  if  $x$  is a linear combination of reduced words starting and ending with a letter in  $\mathcal{M}_i$ , then

COROLLARY 4.8. *If  $E$  is an interpolation space for  $(L_2, L_p)$  with  $2 < p < \infty$ , then for any  $x = \sum_i x_i \in \mathcal{M}$  with  $x_i \in \mathcal{L}_i \cap \mathcal{R}_i$ ,*

$$\|x\|_E \approx_p \left\| (\mathcal{E}(x^*x))^{1/2} \right\|_E + \left\| (\mathcal{E}(xx^*))^{1/2} \right\|_E + \left\| \sum_i x_i \otimes e_i \right\|_{E(\mathcal{M} \overline{\otimes} \ell_\infty)}.$$

One can also get a statement for  $1 < p < 2$ .



#### 4.4. Free Khinchin inequalities for words of length $d$

In this subsection, we explain how to apply the method used above to free Khinchin inequalities for words of fixed length  $d \in \mathbb{N}^*$ . The proof follows the same steps as the one of Theorem 3.10. On one hand, the setting of Khinchin inequalities simplifies the arguments considerably since all the variables are automatically centered and the duality is straightforward. On the other hand, since we consider words of length  $d$ , we cannot avoid to deal with technicalities of combinatorial nature.

Our basic tool, replacing Voiculescu's inequality, is an inequality due to Buchholz [3] and known to Haagerup. Before stating it, let us introduce some notations. Consider the free group with  $n$  generators  $\mathbb{F}_n$  and let  $\{g_i\}_{i \in \llbracket n \rrbracket}$  be a set of generators of  $\mathbb{F}_n$  with  $\llbracket n \rrbracket = \{1, \dots, n\}$ . For any  $d \in \mathbb{N}^*$ , define  $W_d^+ = \{g_{i_1} \dots g_{i_d} : i \in \llbracket n \rrbracket^d\}$  and identify  $W_d^+$  with  $\llbracket n \rrbracket^d$ . Let  $\mathcal{M}$  be finite,  $\mathcal{N} = \mathcal{M} \otimes VN(\mathbb{F}_n)$  and  $\lambda : \mathbb{F}_n \rightarrow VN(\mathbb{F}_n)$  the natural inclusion. Let  $x : W_d^+ \rightarrow \mathcal{M}$ . The Buchholz–Haagerup inequality is expressed in terms of the matrices  $(x_{\alpha, \beta}^{[k]})_{\alpha \in W_{d-k}^+, \beta \in W_k^+}$  in  $\mathbb{M}_{n^k, n^{d-k}}(\mathcal{M})$  defined by

$$x_{\alpha, \beta}^{[k]} = x(\alpha\beta),$$

where  $k$  ranges from 0 to  $d$ , see [19]. We consider only words in the generators (and not their inverses) to avoid cancellations and to make the presentation easier.

It will be useful, in particular, when stating Lemma 4.13, to embed these matrices in a common algebra  $\mathcal{A} := \mathcal{M} \otimes \mathbb{M}_n^{\otimes d}$  with its natural trace  $\tau_{\mathcal{M}} \otimes tr^{\otimes d}$ . For  $i \in \llbracket n \rrbracket$ , denote by  $c_k$  (respectively,  $r_k$ ) the element  $e_{k,1}$  (respectively,  $e_{1,k}$ ) in  $\mathbb{M}_n$ . Define

$$i_k : \mathbb{M}_{n^k, n^{d-k}}(\mathcal{M}) \rightarrow \mathcal{A}$$

$$a = (a_{i,j})_{i \in \llbracket n \rrbracket^{d-k}, j \in \llbracket n \rrbracket^k} \mapsto \sum_{i \in \llbracket n \rrbracket^{d-k}, j \in \llbracket n \rrbracket^k} a_{i,j} \otimes c_{i_1} \otimes \dots \otimes c_{i_{d-k}} \otimes r_{j_1} \otimes \dots \otimes r_{j_k}.$$

Denote by  $\mathcal{A}_k$  the range of  $i_k$ , which is by construction isomorphic to  $\mathbb{M}_{n^{d-k}, n^k}(\mathcal{M})$ . Define

$$G(x) := \sum_{i \in \llbracket n \rrbracket^d} x(i) \otimes \lambda(g_{i_1} \dots g_{i_d}), \text{ and } [x]_k = i_k(x^{[k]}),$$

or more explicitly:

$$[x]_k := \sum_{i \in \llbracket n \rrbracket^d} x(i) \otimes c_{i_1} \otimes \dots \otimes c_{i_{d-k}} \otimes r_{i_{d-k+1}} \otimes \dots \otimes r_{i_d}.$$

We extend those notations to any  $X \in \mathcal{A}_k$  the following way: if  $X = [x]_k$  for some  $x \in F(W_d^+, \mathcal{M})$  and  $k' \in \{0, \dots, d\}$ , then  $[X]_{k'} := [x]_{k'}$  and  $G(X) := G(x)$ . Denote also by  $tr_k : \mathcal{A} \rightarrow \mathcal{M} \otimes \mathbb{M}_n^{\otimes k}$  the operator defined by  $tr_k = Id \otimes tr^{\otimes d-k} \otimes Id^{\otimes k}$ .

Finally, we identify  $\mathcal{M} \otimes \mathbb{M}_n^{\otimes k}$  with  $\mathcal{M} \otimes e_{1,1}^{\otimes d-k} \otimes \mathbb{M}_n^{\otimes k}$  in  $\mathcal{A}$ ; this is a non-unital trace-preserving  $*$ -homomorphism so that for  $e \in \mathcal{M} \otimes \mathbb{M}_n^{\otimes k}$  and  $X \in \mathcal{A}_k$ , we have  $Xe \in \mathcal{A}_k$ .

**THEOREM 4.9** (Buchholz, Haagerup). *For any  $x$  in  $F(W_d^+, \mathcal{M})$ ,*

$$\max_{0 \leq k \leq d} \|[x]_k\|_{\mathcal{A}} \leq \|G(x)\|_{\mathcal{N}} \leq \sum_{k=0}^d \|[x]_k\|_{\mathcal{A}}.$$

**REMARK 4.10.** We will also use the following dual inequality:

$$\|G(x)\|_{L_1(\mathcal{N})} \leq \inf_{0 \leq k \leq d} \|[x]_k\|_{L_1(\mathcal{A})}.$$

*Proof.* Fix  $0 \leq k \leq d$ . Using the polar decomposition, we can decompose  $[x]_k = [y]_k a$  with  $a \in \mathcal{M} \otimes \mathbb{M}_n^{\otimes k}$  such that  $\|[x]_k\|_1 = \|[y]_k\|_2 \|a\|_2$ . Write

$$a = \sum_{j, m \in \llbracket n \rrbracket^k} a_{j, m} \otimes e_{1,1}^{\otimes d-k} \otimes e_{j_1, m_1} \otimes \cdots \otimes e_{j_k, m_k}.$$

Consider now the algebra  $\mathcal{N} \overline{\otimes} V N(\mathbb{F}_k)$  with  $(h_i)_{i \in \llbracket k \rrbracket}$  designating a set of generators for the new copy of  $\mathbb{F}_k$ . Define

$$Y = \sum_{i \in \llbracket n \rrbracket^{d-k}, j \in \llbracket n \rrbracket^k} y_{i,j} \otimes \lambda(g_{i_1} \cdots g_{i_{d-k}} h_{j_1} \cdots h_{j_k}),$$

and

$$A = \sum_{j, m \in \llbracket n \rrbracket^k} a_{j, m} \otimes \lambda(h_{j_k}^{-1} \cdots h_{j_1}^{-1} g_{m_1} \cdots g_{m_k}).$$

Note that if  $\mathbb{E}$  designates the conditional expectation from  $\mathcal{A} \overline{\otimes} V N(\mathbb{F}_\infty)$  to  $\mathcal{A}$ ,  $\mathbb{E}(YA) = G(x)$ . Therefore,  $\|G(x)\|_1 \leq \|Y\|_2 \|A\|_2 = \|y\|_2 \|a\|_2 = \|[x]_k\|_1$ .  $\square$

REMARK 4.11. Dualizing the above remark implies that the norm of the projection in  $\mathcal{N}$  onto words of length  $d$  in the generators is bounded by  $d + 1$ . This is different from the constant  $2d$  that appears in [19] for the projection onto all words of length  $d$ . Of course, this extends to any fully symmetric space using duality and interpolation.

REMARK 4.12. Since we consider only words in  $W_d^+$ , the Buchholz–Haagerup inequality can be improved. Indeed, by [24],

$$\|G(x)\|_{\mathcal{N}} \leq 4^5 \sqrt{e} \left( \sum_{k=0}^d \|[x]_k\|_{\mathcal{A}}^2 \right)^{1/2}.$$

As a consequence, the asymptotic behavior of constants appearing in the rest of this section can be made more precise. For example, the constant  $d + 1$  in Lemma 4.19 can be replaced by  $4^5 \sqrt{e(d+1)}$ .

To simplify some notation if  $\alpha = \sum_j m_j \otimes v_j \in \mathcal{M} \otimes \mathbb{M}_n^{\otimes k}$  with  $m_j \in \mathcal{M}$  and  $v_j \in \mathbb{M}_n^{\otimes k}$ , and  $\beta \in \mathbb{M}_n^{\otimes k'-k}$  with  $k' \geq k$ , we denote by  $\alpha \otimes \beta \in \mathcal{M} \otimes \mathbb{M}_n^{\otimes k'}$  the element  $\sum_j m_j \otimes \beta \otimes v_j$ .

The following lemma can be easily checked by the reader. It is constituted of the algebraic identities that will allow us to bypass the difficulty caused by considering words of length  $d$ .

LEMMA 4.13. *Let  $0 \leq k \leq k' \leq d$ . Let  $a, b : \llbracket n \rrbracket^d \rightarrow \mathcal{M}$  and  $\alpha \in \mathcal{M} \otimes \mathbb{M}_n^{\otimes k}$ . Then,*

- (i)  $\tau_{\mathcal{N}}(G(a)^* G(b)) = \tau_{\mathcal{A}}([a]_k^* [b]_k);$
- (ii)  $[a]_k^* [b]_k = \text{tr}_k([a]_{k'}^* [b]_{k'});$
- (iii)  $[a]_k \alpha = [[a]_{k'} (1_{\mathbb{M}_n}^{\otimes k'-k} \otimes \alpha)]_k;$
- (iv)  $[a]_{k'} (1_{\mathbb{M}_n}^{\otimes k'-k} \otimes \alpha) = [[a]_k \alpha]_{k'}.$

We can now start to reproduce the scheme of proof of the previous sections, starting with proving the existence of an optimal decomposition.

PROPOSITION 4.14. Let  $x \in F(W_d^+, \mathcal{M})$ . There exists  $y_0, \dots, y_d \in F(W_d^+, \mathcal{M})$  such that  $y_0 + \dots + y_d = x$  and

$$\sum_{k=0}^d \|[y_k]_k\|_{L_1(\mathcal{A})} = \inf_{z_0 + \dots + z_d = x} \left\{ \sum_{k=0}^d \|[z_k]_k\|_{L_1(\mathcal{A})} \right\}.$$

As usual, this proposition can be easily obtained once the following lemma is known.

LEMMA 4.15. Let  $x \in F(W_d^+, \mathcal{M})$ . Let  $y_0, \dots, y_d \in F(W_d^+, \mathcal{M})$  such that  $y_0 + \dots + y_d = x$ . Then, there exist  $z_0, \dots, z_d \in F(W_d^+, \mathcal{M})$  such that  $z_0 + \dots + z_d = x$  and for all  $0 \leq k \leq d$ ,  $\|[z_k]_k\|_\infty \leq C_{d,n} \|[x]_0\|_\infty$  and  $\|[z_k]_k\|_1 \leq \|[y_k]_k\|_1$  for some constant  $C_{d,n}$ .

*Proof.* Note that the norms on  $F(W_d^+, \mathcal{M})$  given by  $\|[\cdot]_k\|_\infty$  for  $0 \leq k \leq d$  are all equivalent. We argue by induction, showing that for every  $j \in \{0, \dots, d+1\}$ , we can find  $z_0, \dots, z_d$  such that for every  $k \in \{0, \dots, d\}$ ,  $\|[z_k]_k\|_1 \leq \|[y_k]_k\|_1$  and for every  $k < j$ ,  $\|[z_k]_k\|_\infty \lesssim \|[x]_0\|_\infty$ . Assume that the statement is known for a fixed  $j \in \{0, \dots, d\}$  and let  $w_0, \dots, w_d \in F(W_d^+, \mathcal{M})$  verifying its conditions. Let  $a = x - w_0 - \dots - w_{j-1}$ . By the equivalence of norms mentioned above and by our induction hypothesis,  $A := \|[a]_j\|_\infty \lesssim \|[x]_0\|_\infty$ . Let  $e = 1_{[0,A]}(|[w_j]_j|)$  in  $\mathcal{M} \otimes \mathbb{M}_n^{\otimes j}$ . Define  $z_0, \dots, z_d$  by

$$[z_k]_k = \begin{cases} [w_k]_k, & \text{if } k < j, \\ [w_j]_j e + [a]_j (1_{\mathcal{M} \otimes \mathbb{M}_n^{\otimes j}} - e) & \text{if } k = j, \\ [w_k]_k (1_{\mathbb{M}_n}^{\otimes k-j} \otimes e) & \text{if } k > j. \end{cases}$$

Using Lemma 4.13, it is clear that  $z_0 + \dots + z_d = x$ . The other conditions are verified in a similar way as for Lemma 3.5.  $\square$

An optimal decomposition verifies the algebraic properties expressed in the following proposition.

PROPOSITION 4.16. Let  $x \in F(W_d^+, \mathcal{M})$  and  $y_0, \dots, y_d$  be an optimal decomposition of  $x$ . There exists  $u \in F(W_d^+, \mathcal{M})$  and  $\gamma_k \in (\mathcal{M} \otimes \mathbb{M}_n^{\otimes k})^+$  for all  $k = 0, \dots, d$  such that for all  $0 \leq k \leq d$ ,

$$s(\gamma_k) \leq [u]_k^* [u]_k \leq 1 \text{ in } \mathcal{M} \otimes \mathbb{M}_n^{\otimes k} \text{ and } [y_k]_k = [u]_k \gamma_k.$$

*Proof.* We do not give a detailed proof of this proposition since it is very similar to Proposition 3.4 of this paper or Theorem 4.2. in [4]. The idea is to consider an optimal decomposition  $x = y_0 + \dots + y_d$  given by Proposition 4.14 and then, to write the polar decomposition  $[y_k]_k = v_k \gamma_k$  and argue by duality. Note that for the duality argument, it is convenient to come back to the more standard point of view and identify  $[y_k]_k \in \mathcal{A}_k$  with  $y_k^{[k]} \in \mathbb{M}_{n^k, n^{d-k}}$ .  $\square$

REMARK 4.17. Remark also that above we consider the right modulus of each  $y_k$  even though this is not the natural thing to do for  $y_d$  which is a row. The reason is that it will make the writing of the following proofs smoother.

**THEOREM 4.18.** *Let  $x \in F(W_d^+, \mathcal{M})$  and  $y_0, \dots, y_d$  an optimal decomposition of  $x$ . Let  $E$  be a fully symmetric function space. Then for all  $0 \leq k \leq d$ :*

$$\frac{1}{d+1} \| [y_k]_k \|_{E(\mathcal{A})} \leq \| G(x) \|_{E(\mathcal{N})} \leq (d+1) \sum_{k=0}^d \| [y_k] \|_{E(\mathcal{A})}.$$

**LEMMA 4.19.** *Let  $0 \leq k \leq d$ ,  $u \in F(W_d^+, \mathcal{M})$  and  $\gamma \in (\mathcal{M} \otimes \mathbb{M}_n^{\otimes k})^+$ . Suppose that for every  $0 \leq k' \leq d$ ,  $\| [u]_{k'} \|_\infty \leq 1$  and  $[u]_k^* [u]_k \gamma = \gamma$ . Then, for any fully symmetric space  $E$ ,*

$$\| G([u]_k \gamma) \|_{E(\mathcal{N})} \leq (d+1) \| \gamma \|_{E(\mathcal{A})}.$$

*Proof.* Since  $E$  is fully symmetric, it suffices to prove the lemma for  $E$  of the form  $L_1 + tL_\infty$ ,  $t > 0$ . Then, by fixing a decomposition of  $\gamma$  into two elements which can be taken to be non-negative, we are reduced to treating the cases of  $L_1$  and  $L_\infty$ . By Remark 4.10, the case of  $L_1$  is straightforward. To treat the case of  $L_\infty$ , we use the Buchholz–Haagerup inequality. We need to prove that, for all  $0 \leq k' \leq d$ ,

$$\| [[u]_k \gamma]_{k'} \|_\infty \leq \| \gamma \|_\infty.$$

Do to this, we use Lemma 4.13. If  $k' \geq k$ , we use (iv):

$$\| [[u]_k \gamma]_{k'} \|_\infty = \left\| [u]_{k'} (1_{\mathbb{M}_n}^{\otimes k'-k} \otimes \gamma) \right\|_\infty \leq \| [u]_{k'} \|_\infty \| \gamma \|_\infty \leq \| \gamma \|_\infty.$$

If  $k' < k$ , recall that  $[u]_k^* [u]_k \gamma = \gamma$ :

$$[[u]_k \gamma]_{k'}^* [[u]_k \gamma]_{k'} = \text{tr}_{k'}(\gamma [u]_k^* [u]_k \gamma) = \text{tr}_{k'}([u]_k^* [u]_k \gamma [u]_k^* [u]_k \gamma [u]_k^* [u]_k).$$

Now note that  $\| [u]_k \gamma [u]_k^* [u]_k \gamma [u]_k^* \|_\infty \leq \| \gamma^2 \|_\infty$ . Hence

$$[[u]_k \gamma]_{k'}^* [[u]_k \gamma]_{k'} \leq \| \gamma^2 \|_\infty \text{tr}_{k'}([u]_k [u]_k^*) = \| \gamma^2 \|_\infty [u]_{k'}^* [u]_{k'} \leq \| \gamma^2 \|_\infty. \quad \square$$

*Proof of Theorem 4.18.* The second inequality is straightforward by the triangle inequality and Lemma 4.19 so let us focus on the first one. Once again, it suffices to show it for  $E = L_1 + tL_\infty$ ,  $t > 0$ . We argue by duality. Fix  $0 \leq k \leq d$ . There exists a spectral projection  $e$  of  $\gamma_k$  in  $\mathcal{M} \otimes \mathbb{M}_n^{\otimes k}$  such that  $\tau(e) = t$  and  $\tau_{\mathcal{A}}(e\gamma_k) = \| \gamma_k \|_{L_1 + tL_\infty}$ . By Lemma 4.19,  $\| G([u]_k e) \|_{L_\infty \cap t^{-1}L_1} \leq d+1$ . Let us compute  $\tau_{\mathcal{N}}(G([u]_k e)^* G(x))$ , relying once again heavily on Lemma 4.13 and recalling that  $e[u]_k^* [u]_k = e$  and  $[u]_j^* [u]_j \gamma_j = \gamma_j$  for any  $j \leq d$ .

$$\begin{aligned} \tau_{\mathcal{N}}(G([u]_k e)^* G(x)) &= \sum_{j=0}^d \tau_{\mathcal{A}}([u]_k e)^* [y_j]_k) \\ &= \sum_{j \leq k} \tau_{\mathcal{A}}(e [u]_k^* [[u]_j \gamma_j]_k) + \sum_{j > k} \tau_{\mathcal{A}}([e [u]_k^*]_j [u]_j \gamma_j) \\ &= \sum_{j \leq k} \tau_{\mathcal{A}}(e [u]_k^* [u]_k (1_{\mathbb{M}_n}^{\otimes k-j} \otimes \gamma_j)) + \sum_{j > k} \tau_{\mathcal{A}}((1_{\mathbb{M}_n}^{\otimes j-k} \otimes e) [u]_j^* [u]_j \gamma_j) \\ &= \sum_{j \leq k} \tau_{\mathcal{A}}(e (1_{\mathbb{M}_n}^{\otimes k-j} \otimes \gamma_j)) + \sum_{j > k} \tau_{\mathcal{A}}((1_{\mathbb{M}_n}^{\otimes j-k} \otimes e) \gamma_j) \\ &\geq \tau_{\mathcal{A}}(e \gamma_k) = \| \gamma_k \|_{L_1 + tL_\infty}. \end{aligned}$$

Hence,  $(d+1) \| Gx \|_{L_1 + tL_\infty} \geq \| \gamma_k \|_{L_1 + tL_\infty}$ . □

COROLLARY 4.20. Let  $x \in F(W_d^+, \mathcal{M})$  and  $E$  be a fully symmetric space. Then:

$$\frac{1}{(d+1)^2} \inf_{y_0+\dots+y_d=x} \left\{ \sum_{k=0}^d \|[y_k]_k\|_E \right\} \leq \|G(x)\|_E \leq (d+1)^2 \max_{0 \leq k \leq d} \|[x]_d\|_E.$$

Moreover, if  $E$  is an  $(L_1, L_2)$ -interpolation space, then

$$\|G(x)\|_E \leq \inf_{y_0+\dots+y_d=x} \left\{ \sum_{k=0}^d \|[y_k]_k\|_E \right\},$$

and if  $E$  is an  $(L_2, L_\infty)$ -interpolation space, then

$$\max_{0 \leq k \leq d} \|[x]_d\|_E \leq (d+1) \|G(x)\|_E.$$

*Proof. Main inequality.* The left-hand side is a direct consequence of Theorem 4.18. For the right-hand side, we also use Theorem 4.18 together with the fact that for any  $k \in \{0, \dots, d\}$ ,  $\|[y_k]_k\|_E \leq \|[x]_k\|_E$ . To prove this latter claim, we rely once again on the same technique. It suffices to prove it for  $E = L_1 + tL_\infty$ . By duality, there exists an element  $f \in \mathcal{A}$  such that  $\tau(f[y_k]_k) = \|[y_k]_k\|_{L_1+tL_\infty}$ ,  $\|f\|_1 \leq t$  and  $\|f\|_\infty \leq 1$ . Furthermore,  $f$  can be chosen of the form  $e[u]_k^*$  where  $e$  is a projection in  $\mathcal{M} \otimes \mathbb{M}_n^{\otimes k}$ . Repeating the computation above, by Lemma 4.13:

$$\begin{aligned} \|[x]_k\|_{L_1+tL_\infty} &\geq \tau_{\mathcal{A}}(e[u]_k^*[x]_k) = \sum_{j \leq k} \tau_{\mathcal{A}}(e[u]_k^*[[u]_j \gamma_j]_k) + \sum_{j > k} \tau_{\mathcal{A}}([e[u]_k^*]_j [u]_j \gamma_j) \\ &\geq \tau_{\mathcal{A}}(e \gamma_k) = \|y_k\|_{L_1+tL_\infty}. \end{aligned}$$

If  $E$  is an  $(L_1, L_2)$ -interpolation space. First, note that by Remark 4.10 and interpolation, for any  $k \in \{0, \dots, d\}$ ,  $\|Gx\|_E \leq \|[x]_k\|_E$ . Using the triangle inequality, we obtain the desired estimate.

If  $E$  is an  $(L_2, L_\infty)$ -interpolation space. This inequality follows from the previous one by duality. The argument necessitates to use the boundedness of the projection on words of length  $d$  given by Remark 4.11.  $\square$

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